Notes on Quantum Mechanics
Lectures by Prof. Barton Zwiebach
MIT OCW Physics 8.05

Herein find notes from Barton Zwiebach lectures on Quantum Mechanics, Physics 8.05 in the MIT Opencourseware catalog. Full set of notes is available on the MIT OCW course site, and they're really complete and helpful. These here are just the main new ideas I've learned. Zwiebach is extraordinary, and he's clarified a whole bunch of concepts for me.

Lecture 1

At first, I thought Oh, no. Here's another dull presentation of the wavefunction. My mind goes numb. At least it has in past efforts to figure out QM.

This is different. Zwiebach is so enthusiastic and so clear it actually begins to make sense. Here are the key ideas.

Start with the Schrodinger general equation.

$$\mathrm{i}\hbar \frac{\partial \Psi(x,t)}{\partial \mathrm{t}} = \left(\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x,t) \right) \Psi(x,t)$$

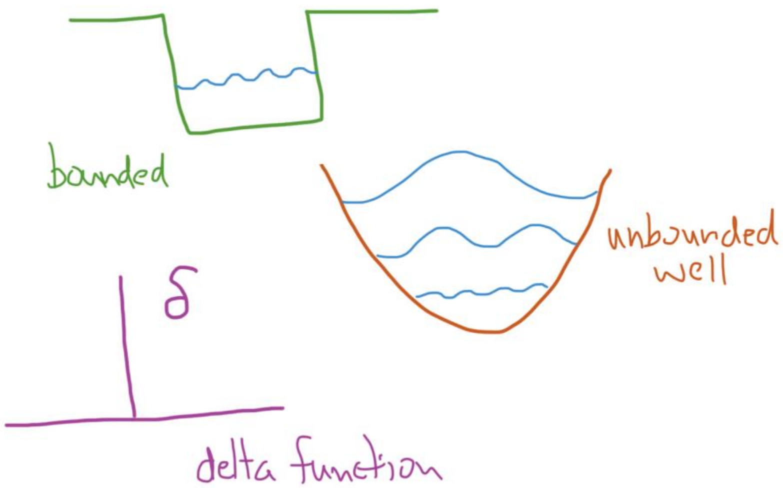
Note the symbols. Capital Psi refers specifically to the general equation. We'll see later that little psi refers to the time-independent equation

$$\widehat{H}\psi(x) = E\psi(x)$$

Those parentheses on the right of the general equation by the way. That's $\widehat{H}$, the time operator Hamiltonian. The total energy. That's where it comes from. How come I never realized that before?

Allowed potentials include, among the more common:

Constraints on $V(x)$



Develop the mathematical tools. I'll drop the function parameters, for clarity. But keep in mind Ψ means $\Psi(x,t)$, while ψ means $\psi(x)$.

If Ψ represents a particle, we need maths to locate the particle in space and time. Ψ is complex. It has to be, given that i on the l.h.s. of the general equation. In order to locate particles in the real world we need real values. Logical choice is the metric from complex math. The density function

$$\rho(x,t) \equiv \Psi^*\Psi$$

Location first. Main idea, as usual for unitarity in probability, is that the particle exists, with probability one, so we must be able to find it somewhere. In one dimension, the particle has to be somewhere along the real number line in the range minus infinity to plus infinity. So

$$\int_{-\infty}^{+\infty} \Psi^*\Psi \, dx = 1$$

Note that in the integral, $\Psi^*\Psi \, dx$ is the probability of finding the particle somewhere in the interval $(x, x+dx)$.

All that is familiar. Unitary probability. Amplitude vs. probability. New is better understanding of the density function. I can sort of see it now; there it is on the Real line.

Zwiebach makes a point on conditions at infinity. Both Psi and $\frac{\partial\Psi}{\partial x}$ have to go to zero at $\pm\infty$. Otherwise probability and momentum, among other things, blow up.

Next is the continuity equation for the wavefunction. Here's a puzzlement. It's easy to visualize charge conservation, for example, in the continuity equation

$$\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0$$

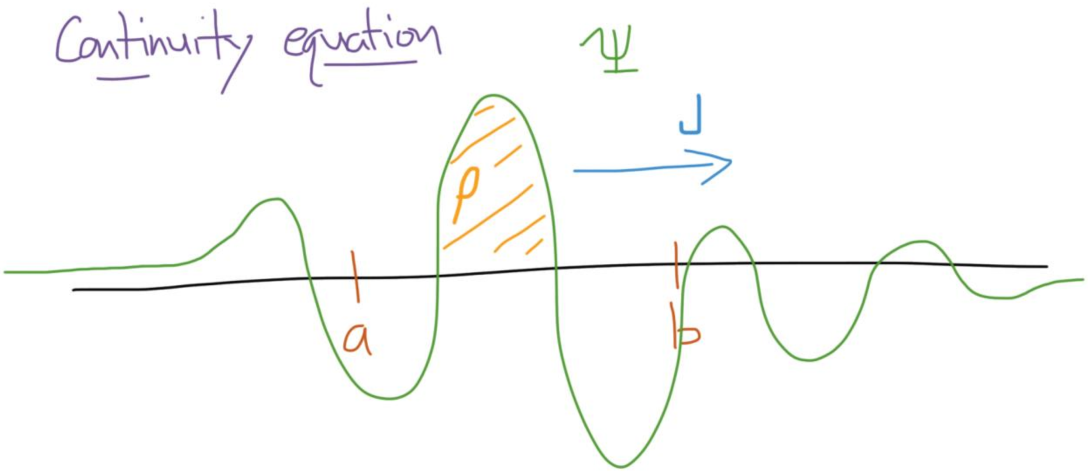
Any charge that escapes a region of space must have passed through the boundary of that region. In a one-dimensional system

$$\frac{\partial J}{\partial x} + \frac{\partial \rho}{\partial t} = 0$$

This makes sense in Zwiebach's example. Consider an interval a to b on the real line. Any change in the density of the wavefunction in that interval must result from a density current.

$$d\rho = J(a) - J(b)$$

given the sign convention that J increases to the right.



What is it that's being conserved here? Well, I think it's conservation of probability. If amplitude shifts in space – if the wavefunction is wiggling or a wave packet traveling – then the probability to find a particle must shift from one region to another. Probability of finding it somewhere remains 1, so the probability density has to increase in the neighboring interval if it decreases in this interval right here.

This feels good, finally getting a handle on the ideas! At least I think I am . . .

Next up, the operator $\widehat{H}$. In general, operators change the wavefunction. $\widehat{H}$ is the time step operator. If the wavefunction is time dependent, e.g. if the wave is oscillating, then $\widehat{H}$ updates the wavefunction to the next time step.

Here's the eye opener I mentioned at the beginning. Take the general Schrodinger equation.

$$i\hbar\frac{\partial\Psi(x,t)}{\partial \mathrm{t}} = \left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right)\Psi(x,t)$$

Remove the time dependence, i.e. consider wavefunction in a fixed potential and psi itself a function (remember we're talking mathematical functions) depending only on position as the independent variable. Rewrite.

$$\mathrm{i}\hbar\frac{\partial}{\partial\mathrm{t}}\psi(x) = \left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x)$$

Parentheses on the right represents total energy. Relabel the differential operator on the left and, voila

$$\hat{H}\psi(x) = E\psi(x)$$

Note that $\hat{H}$ is an operator and E is real. So this is an eigenvector / eigenvalue expression. We'll get to that in a minute; we can solve the equation to find energy eigenvalues of the wavefunction, i.e. we can find the energy spectrum of a quantum system. First solve the differential equation. It is first order in time, so pretty straightforward.

$$\mathrm{i}\hbar\frac{\partial}{\partial\mathrm{t}}\psi(x) = E\psi(x)$$

$$\psi(x,t) = e^{-iEt/\hbar}\,\psi(x)$$

Neat! We can calculate how the wavefunction evolves over time. Draw it in space. Cartoon animate it to watch it change over time. Update by that exponential in energy.

On to energy eigenstates. The wavefunction will have particular energy solutions, $b_n\psi_n$ which are basis states in a vector space. So a general wavefunction can be expressed as

$$\psi = \sum_{i=1}^{n} b_i\psi_i$$

where n is the number of eigenstates, i.e. energy solutions. The summation above is the spectrum of the wavefunction, e.g. the energy states of a hydrogen atom. Interesting physics occurs in degenerate states, when more than two eigenstates have the same energy. Zwiebach promises further explanation . . .

The eigenstates are orthonormal, as expected in linear algebra. In mathspeak

$$\int \psi_m\psi_n\,dx = \delta_{mn}$$

All this is practically useful for calculating the general wavefunction

$$\Psi(x,t) = \sum_{j=1}^{n} b_j e^{-iEt/\hbar}\, \psi_j$$

and for calculating the coefficients of the eigenstates

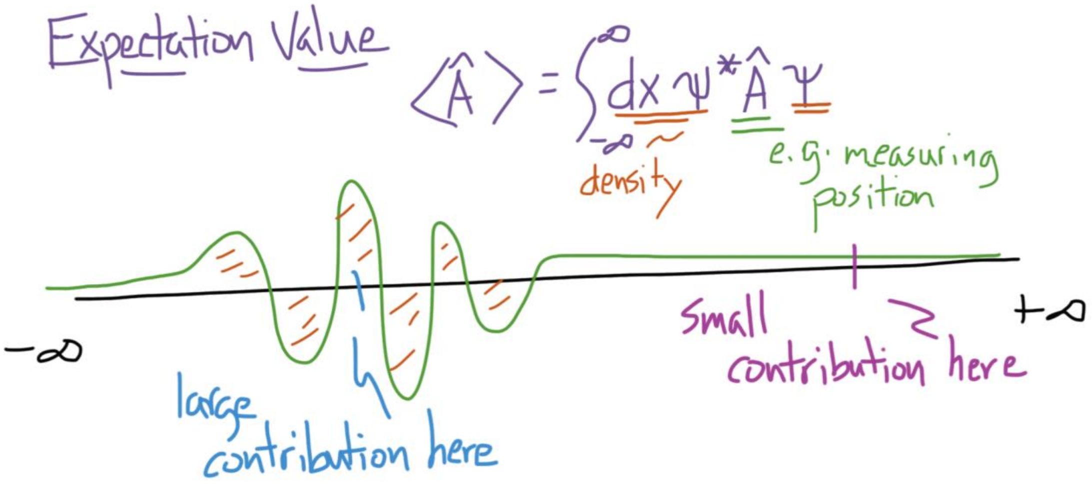
$$b_m = \int dx\, {\psi_m}^{*} \psi$$

Pay attention what's going on. The summation above shows how to find the general wavefunction from the stationary function. All the time dependence is in the energy exponential. All the evolution, all the dynamics is in that energy function. The integral, finding coefficients, is all about orthogonality. Dot product (this really is just a dot product, in integral form) picks out the term in question. By orthogonality, all other products go to zero.

Finally, expectation value. Given a general time-independent operator, $\hat{A}$, what value can we expect on repeated / averaged measurements? Here 'tis.

$$\langle \hat{A} \rangle_{\Psi}(t) = \int_{-\infty}^{+\infty} \Psi^{*} \hat{A} \Psi \, dx$$

Real value on the left. Functions on the right. So the integral is a functional, converting a function to a number.



Think about that a minute. Suppose you're trying to find a particle's position. Looking for the expectation value of the position operator. Well, that argument in the integral is teasing out the likeliest position from the probability density, the product of those Psi's. The integral is finding average value over an infinite range where total probability is one, so there's no need for the usual $1/(b-a)$ coefficient out front. Particle is most likely to be found where the probability density is greatest.

And while I'm at it, I finally get the notion why $\psi^*(\hat{A}\,\psi)$ gives an average value for the operator. The operator transforms the wavefunction. That's what matrices / operators do. Rotate or stretch vectors. Assuming the wavefunction is normalized, then the dot product of the vector with its transformed self gives you the projection, how much of that wavefunction you can expect to find with that observation. Projections on vectors. Observables and how much you can expect to observe.

eigenstates, spectrum, and degeneracy

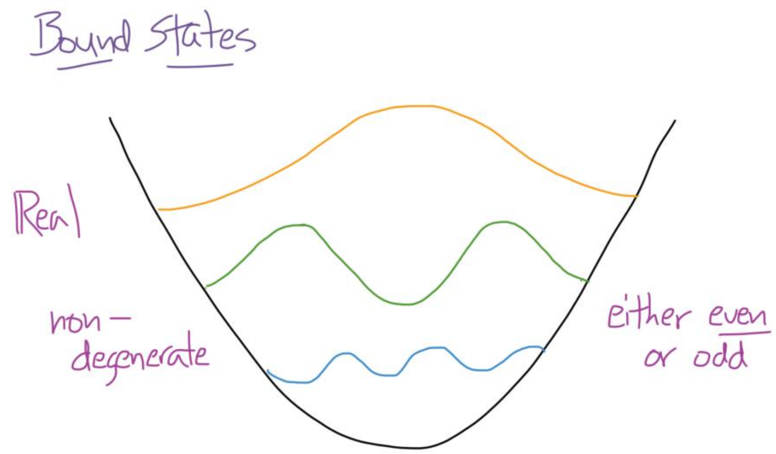
Lecture 2

Starts out with Theorems about bound states, i.e. states that go to zero at $\pm\infty$.

They're non-degenerate. No duplications of states at the same energy.

They're real.

And they're either even or odd functions.



That's a real brief summary of Zwiebach's presentation, which includes some corollaries and strategies for the proofs. See the course notes for details.

Lecture 3

Position and momentum. Observables. And we can think of them also as bases. Zwiebach introduces the essentials of linear algebra.

Main take-aways are that you can switch from one basis to another by the Fourier transforms.

$$\psi(x) = \int_{-\infty}^{\infty} dp\, e^{ipx/\hbar}\, \tilde{\psi}(p)$$

and

$$\tilde{\psi}(p) = \int_{-\infty}^{\infty} dx\, e^{-ipx/\hbar}\, \psi(x)$$

Note that the momentum operator acting on the x-basis wavefunction gives the proper eigenvalue relations, and vice versa for the p-basis wavefunction.

$$\int_{-\infty}^{\infty} dp\, e^{ipx/\hbar}\, \tilde{\psi}(p) \cong \sum_{j=1}^{N} e^{ip_j x/\hbar}\, \tilde{\psi}(p_j)$$

so

$$\hat{p}\psi(x) = -i\hbar\frac{\partial}{\partial x}\psi(x) = -i\hbar\frac{\partial}{\partial x}\sum_{j=1}^{N} e^{ip_j x/\hbar}\, \tilde{\psi}(p_j) = p\psi(x)$$

as expected. Note the role of that exponential.

One of the neat things I learned in this lecture is how to think of the wavefunction as a vector. Draw a one-dimensional $\psi(x)$. Parse out the function over intervals ϵ . The wavefunction has a value at each interval. Voila! A vector!

$$\psi(x) = \begin{bmatrix} \psi(0) \\ \psi(\epsilon) \\ \psi(2\epsilon) \\ \psi(3\epsilon) \\ \vdots \end{bmatrix}$$

figure wavefunction as vector

And the position operator, a matrix. Given

$$\hat{x}\psi(x) = x\psi(x)$$

Translate to linear algebra

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & 2\epsilon & \dots \\ \vdots & 0 & 0 & 3\epsilon \end{bmatrix} \begin{bmatrix} \psi(0) \\ \psi(\epsilon) \\ \psi(2\epsilon) \\ \psi(3\epsilon) \\ \vdots \end{bmatrix} = x\psi(x)$$

Makes sense! And now I appreciate why the wavefunctions sit in such a huge (Hilbert) vector space!

Rest of the lecture introduces the Stern-Gerlach experiment. I'll get there in a minute. Key is understanding magnetic moment and how a divergent external B field can separate spin-up from spin-down.

Lectures 4–7: Linear algebra

I learned a whole bunch here. Zwiebach was away for a couple lectures, so Aram Harrow and William Detbold filled in. They zoomed through all the essentials. What a great review of all the essential ideas!

They based their presentation on Axler's text, *Linear Algebra Done Right*. I believe that title. Here are the main take-aways and what now makes sense that didn't before (or that I just assumed I understood but really didn't).

It's all about vector spaces and their properties. Components fields and vectors. Properties CANNDII. Fields for our practical purposes are the reals and complex numbers. Vectors are of many sorts: polynomials, lists, etc. (hmm . . . need to review this.) Addition of vectors commutes. Vector addition is associative. There is a null (zero) vector for multiplication. There is a negative (inverse) for addition. Multiplication is distributive. And identities exist for both addition and multiplication.

Hilbert space is a complex vector space that includes an inner product. Inner product is an operation that gives a field value when two vectors are multiplied. Best example is the dot product. All physics occurs in Hilbert space. I think the physical implication here is that physical space requires a measure of distance, and that's given by the inner product.

Subspaces. Need to understand them especially with regard to basis. Here's the link.

A subspace U of vector space V contains the field and vectors $\vec{u}$ such that $U \subset V$ and U itself is a vector space by all the definitions. Now it must be the case for some combination of $U's$ that

$$U_1 \oplus U_2 \oplus . . . U_n = V$$

where the $\oplus$ indicate what's called the 'direct sum.' Regular ol' addition, but we're building a vector space.

Now consider. Suppose each U comprises linearly independent basis vectors that span U . Then the direct sum spans V. A basis for V. And that's the definition of a basis: a set of linearly independent vectors that spans the vector space.

Operators are maps that transform vectors in a vector space V to other vectors in that same vector space. Operators can be represented by matrices; matrices are more narrowly defined as operators in a given basis. i.e. the matrices can be defined by how they transform the vectors in the chosen basis. Operators are functions and obey the distributive laws and linearity, but they do not necessarily commute. The commutation part is interesting; the commutator is a kind of eigenvalue relation. Suppose operator $R = x$ and operator $S = {}^{\partial}/_{\partial x}$. Then

$$[R, S] = Ix$$

I is like an eigval here. Just like $i\hbar$ acts like an eigval in $[x, p] = i\hbar$

Dimension of a vector space, fundamental theorem,

$$\dim V = \dim null(T) + \dim range(T)$$

$null(T)$ are all those vectors that the operator T takes to zero (zero vector). $range(T)$ are all the (surjective) transformations Tv otherwise filling the vector space V . Injective means an operator maps one-to-one. Surjective means the map fills the whole vector space.

Eigens. Eigenvectors and eigenvalues an operator are in the subspace of a vector space such that the operator acting on an eigvec returns another vector in that subspace.

$$T\vec{u} = \lambda\vec{u}$$

Seen that before. But it makes more sense in the context of the spaces.

Which leads to another useful theorem / perspective on the spaces. The eigen subspace plus the subspace orthogonal to the eigens fills the next higher dimension.

$$V = U + U^{\perp}$$

See that with the $x - y$ plane, spanned by e_x and e_y . Add e_z and you've got $\mathbb{R}^3$.

Maybe the biggest 'aha' of these lectures was explanation of Dirac's notation. It all comes out of the inner product in complex space. The ket is a good ol' regular vector. (Remember, whether a vector space is real or complex depends on the field, not the vectors.) The bra on the other hand

(and this was the 'aha') is a map. It maps the ket to a complex value in the field. (That's what happens with an inner product.)

The bra's compose a dual space to the kets, and injective. i.e. the bra is unique to the dual ket. Operations you have to be careful. Apply the rules for complex.

$$\langle a|b\rangle = \langle b|a\rangle^*$$

$$\langle a|\beta_1 b_1 + \beta_2 b_2\rangle = \langle \beta_1 b_1 + \beta_2 b_2|a\rangle^* = \langle \beta_1 b_1|a\rangle^* + \langle \beta_2 b_2|a\rangle^* = \beta_1{}^*\langle a|b_1\rangle + \beta_2{}^*\langle a|b_2\rangle$$

I skipped a step at the last, swapping bra and ket again. That's why the complex star disappears on the last brackets. But you get the idea.

Of course, as usual, you can represent the bra's as row vectors; inner product then just becomes row times column vector multiplication. Handy!

References are in the 8.05 notes, all you need to know, and of course in Axler. What great resources those are.

Detbold and Harrow look like kids, but they are on the frontiers themselves. Harrow works in quantum computers and information theory. He's a student of Chuang's. Detbold runs numerical simulations of strong interactions, including conditions at the core of neutron stars. Neat stuff!

<u>Lecture 8</u>: bra's and ket's and operators

Now we get to the really powerful stuff. Look at what the bra and ket logic can do.

Take for example the definition of the dagger operator and its operations. A couple really useful theorems pop out.

$$\langle T^\dagger u|v\rangle = \langle u|Tv\rangle$$

Note a couple things right off. If we're in a real vector space, then $T^\dagger = (T^*)'$
That's one of the theorems. And you can use it both ways – find the T's if you're in a real vector space or prove you're working with a real vector space if you know the T's. Also note – just formalism stuff – that $\langle u|Tv\rangle = \langle u|T|v\rangle$ and, most important for practical purposes, we can treat the T's as matrices. The other great theorem, by the way

$$\left\langle T^{\dagger}u \middle| v\right\rangle = \langle v|Tu\rangle^{*}$$

Proofs for all these rest on index manipulation. I had it all clear in my head, but then it went fuzzy. I think it's the weather.

Anyway, them's important operators, the T's. I think they'll have a lot to do with, for example, projections and measurement. Expectation values. Such truck.

Lecture 9: bra's and kets (cont'd)

Ha! Told you so. That there expression $\langle u|Tv\rangle = \langle u|T|v\rangle$ is an expectation value! Zwiebach let it slip in passing.

Main arguments in this lecture are extensions of the definition of Hermitian operator. Z makes clear that $T^{\dagger}$ is the adjoint, as defined by the bracket relations above, while $T^{\dagger} = T$ defines Hermitian adjoint. And for Hermitian operators all kinds of wonderful properties follow. For example, the eigenvalues of Hermitian operators must be real. So they can describe measurements. And the eigenvectors of different eigenvalues must be orthogonal. Proofs are pretty slick.

Given T Hermitian and $Tv = \lambda v$ eigens. Start with bracket $\langle v|Tv\rangle$ and work both directions.

$$\lambda\langle v|v\rangle = \langle v|\lambda v\rangle = \langle v|Tv\rangle = \left\langle T^{\dagger}v \middle| v\right\rangle = \langle Tv|v\rangle = \langle \lambda v|v\rangle = \lambda^{*}\langle v|v\rangle$$

Expressions on the two ends are equal, so the lambdas must be real for $\lambda = \lambda^{*}$. Done! All that work with the operators on vector space finding $T^{\dagger}$ pays off!

What about orthogonality of eigenvectors? Here 'tis. Same kind of math manipulation.

Given two different eigenvalues on the same Hermitian operator.

$$\lambda_1\langle v_1|v_2\rangle = \langle \lambda_1 v_1|v_2\rangle = \langle Tv_1|v_2\rangle = \langle v_1|Tv_2\rangle = \langle v_1|\lambda_2 v_2\rangle = \lambda_2\langle v_1|v_2\rangle$$

So $(\lambda_1 - \lambda_2)\langle v_1|v_2\rangle = 0$

But we've stipulated the eigenvalues are different, so $\langle v_1|v_2\rangle = 0$. The eigenvectors must be orthogonal!

Onward to unitary operators. By definition a unitary operator is Hermitian and preserves the norm of any vector in the vector space.

$$|Uv| = |v|$$

From this, since U is Hermitian,

$$\left|U^{\dagger}Uv\right| = |Uv| = |v|$$

So $U^{\dagger}U = I$. The identity. We'll be using that.

Whew! On to bra's. Now here's new. We've been used to

$$|av\rangle = a|v\rangle$$

Fine and dandy. Linearity on a vector space. But look what happens when we start talking about position and momentum. Non-determinate variables. i.e. position, a particle state, could be anywhere in an infinite dimensional Hilbert space. Proper interpretation of the bra, then, is that it represents the state of a particle, not the vector position of the particle. $|x\rangle$ represents a particle at position x . It is the state of the particle being at x . So, now get this,

$$|ax\rangle \neq a|x\rangle$$

Left side is a particle in the state of being at position ax . Right side a is the amplitude of a particle in the state of being at position x . There's a difference! Similarly $|\vec{x}\rangle$ is the state in which a particle is sitting at 3-d vector position $\vec{x}$. The bra itself is not the vector coordinate position. Subtle, eh? But make sense when you think about it.

There's a difference, though, between the meaning of the Dirac notation and the mathematical vector space bracket notation. See if it extends beyond position and momentum. I bet we'll find out soon.

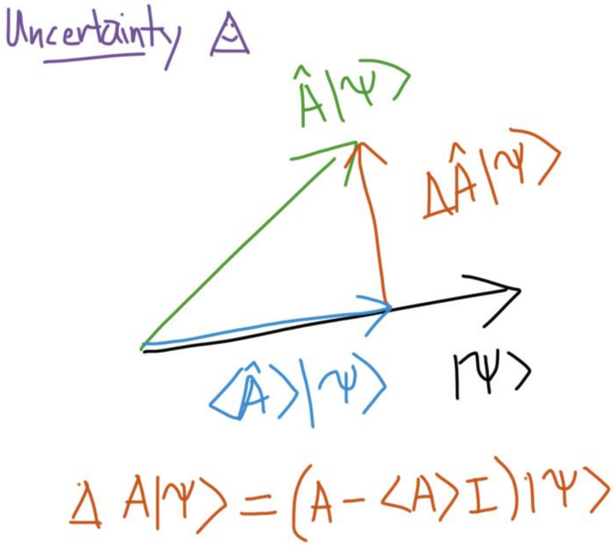
Lecture 10: Uncertainty

Here's the absolute most coolest thing ever. Pythagoras rules!

Question is: how do you figure the uncertainty in an operator? You're trying to find out the value of an observable. Operate on the wavefunction. How certain are you of the result?

Along the way we'll see the importance of the projection operator. But I get ahead of myself.

Here goes. Just put it on the paper, slam bang. Here's how to find the uncertainty of a unitary operator $\hat{A}$ acting on a normalized wavefunction ψ .



A rotates $|\psi\rangle$. First calculate the projection of $A|\psi\rangle$ onto the eigenvector $|\psi\rangle$. That will give the amplitude to find the transformed vector in that eigenstate.

Projection. Here's the idea. It's an operator. Casts the shadow of a state vector onto a basis vector, thereby represents component of that basis in that particular state. And it's simple. Using the $|\psi\rangle$ basis as an example,

$$P_{|\psi\rangle} = |\psi\rangle\langle\psi|$$

That's it! Note it's a matrix, so a likely operator sure enough. And look at the wonders here. Projection of $A|\psi\rangle$ onto the basis

$$|\psi\rangle\langle\psi|A|\psi\rangle = |\psi\rangle\langle A\rangle = \langle A\rangle|\psi\rangle$$

All that's legal because the expectation value $\langle A\rangle$ is a real number (because the operator is Hermitian).

Good enough. Expectation value gives the amplitude of the transformed vector on that basis.

And, methinks, that there explains a lot! Look back at the Dirac definition of the wavefunction.

$$\psi = \int dx\ |x\rangle\langle x|\psi\rangle$$

Well, golly. That's the projection of psi onto the x-basis. Sum over all x and you've got the state vector / wavefunction! Remember, x is a non-determinable basis. It's the Hilbert space of all x. Each position itself serves as a basis. So the definition says the wavefunction is the sum of all the projections of the state vector onto all the bases. Same idea as a vector in three-space is the sum of its $x, y,$ and z components. Makes sense!

Projections onto basis states. Components. Keep that in mind.

But back to the uncertainty. Almost done. Look at the triangle. Because all the vectors are unit vectors, projection is cosine of the angle between $A|\psi\rangle$ and $|\psi\rangle$. Previous of Zwiebach's calculations found that the uncertainty

$$\Delta A|\psi\rangle = A|\psi\rangle - \langle A\rangle I|\psi\rangle$$

Have to have the I in there because we're dealing with matrix operators. But take a look at the figure. $\Delta A|\psi\rangle$ is the sine leg of that Pythagorean triangle. That's it! Vectors!

And Pythagoras immediately gives us the most useful relation for calculating uncertainties.

$$(\Delta A)^2 \geq \langle A^2\rangle - \langle A\rangle^2$$

It's all from the triangle. But the $\langle A^2\rangle$ isn't obvious. Here's the derivation. Remember the state vector is imaginary. So

$$(A|\psi\rangle)^2 = (\langle\psi|A)(A|\psi\rangle)) = \langle\psi|A^2|\psi\rangle = \langle A^2\rangle$$

Pretty neat! There's a whole lot more algebra in the formal proof, but it's all right there in that vector diagram. And a whole lot easier to remember.

Note also the relation to statistics. That 'mean of the square minus the square of the mean' is the statistical variance. Variance. How widely dispersed are the elements in the sample. How much variation is there among the heights of all fifth grade girls. Applied to operators and states, I suppose it could represent the variation you'd expect if you performed a whole series of experiments on particles all prepared in the same initial state. Nice opportunity for a seque here . . .

Lecture 10: the uncertainty principle

Variance. Uncertainty. Related by experiment. The uncertainty in repeatable experiments is the variance around the mean, the expectation value. Measure spin along the *x*-axis of zillions of electrons prepared with spin $|z+\rangle \equiv |+\rangle = |\uparrow\rangle$ and we want to know the variance of our experimental results, our observations, our observable. What is the uncertainty in s_x ?

Think about that again. Prepare spin up along z . Measure along x . Good ol' Stern-Gerlach. Electron spin angular momentum is $\hbar/_2$. Expectation value is zero (because it's equally likely we'll measure $|x+\rangle$ or $|x-\rangle$. We want to know how much variance there will be around the expectation value. Well, duh. Spin is quantized. There are only two possible outcomes to our measurements, $+\hbar/_2$ or $-\hbar/_2$. Variance is $\hbar/_2$. Uncertainty of the outcome is $\hbar/_2$. Uncertainty in the observable s_x is $\hbar/_2$. Done.

Let's see if the math works. We want to find the variance in s_x given state $|z+\rangle$. Use the difference of means / expectation values formula.

$$(\Delta s_x)^2 \geq \langle {s_x}^2 \rangle - \langle s_x \rangle^2$$

Use matrix representation to calculate those expectation values.

$$\langle s_x \rangle = \langle + | s_x | + \rangle = [1 \quad 0] \, \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

$$\langle {s_x}^2 \rangle = \langle + | {s_x}^2 | + \rangle = [1 \quad 0] \left(\frac{\hbar}{2} \right)^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left(\frac{\hbar}{2} \right)^2$$

Putting it all together,

$$(\Delta s_x)^2 \geq \left(\frac{\hbar}{2} \right)^2$$

$$\Delta s_x \geq \frac{\hbar}{2}$$

As expected! The logic works. Variance = uncertainty lower bound as expected from the quantization of spin.

Okay. How about those famous uncertainty principles? $\Delta x \Delta p \geq \frac{\hbar}{2}$ and $\Delta E \Delta t \geq \frac{\hbar}{2}$? Where did they come from? How can we understand them? And how can we figure out the minimum of uncertainties, the minimum variance in our experimental results?

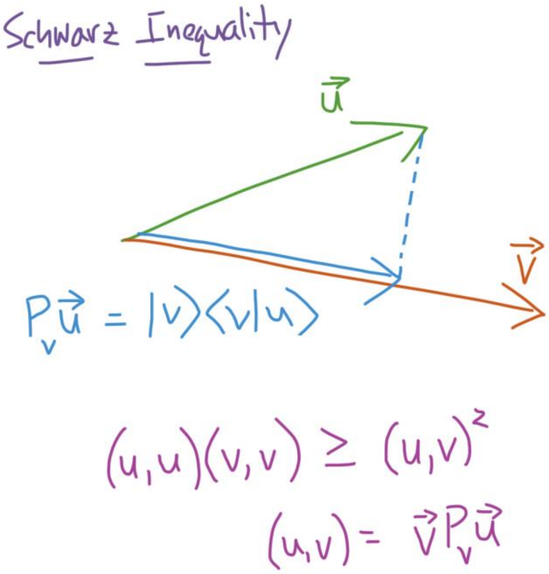
Short answer is

$$\Delta A \Delta B \geq \left(\langle\psi| \, \frac{1}{2i} [A, B] \, |\psi\rangle \right)$$

where A and B are Hermitian operators. Check that out with the known

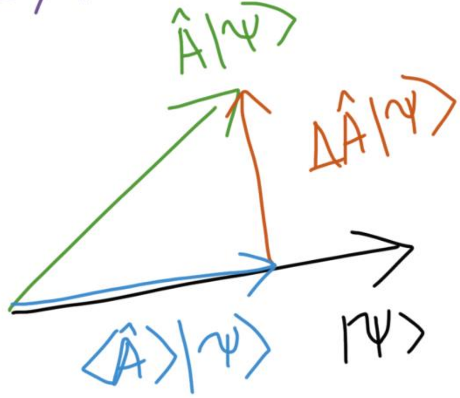
$$\Delta x \Delta p \geq \left(\langle\psi| \, \frac{1}{2i} [x, p] \, |\psi\rangle \right) = \left(\langle\psi| \, \frac{1}{2i} i\hbar \, |\psi\rangle \right) = \frac{\hbar}{2}$$

As expected. Note that the uncertainty is real. Swell. Now where did that come from? Back to basics. We need to figure out that commutator in the expectation value. Who dat?
Begin with the Schwarz inequality. Vector stuff. Product of norms of the two sides of a triangle are always greater than or equal to the product of the projection of a vector and its projector.



Apply that to our geometric interpretation of the uncertainty.

Uncertainty Δ



$$\Delta A|\psi\rangle = (A - \langle A\rangle I)|\psi\rangle$$

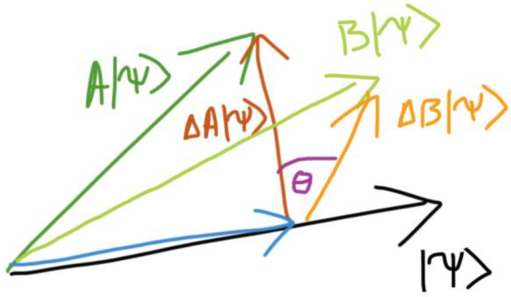
$$\Delta A|\psi\rangle = A|\psi\rangle - \langle A\rangle I|\psi\rangle$$

Add a second uncertainty to the mix.

$$\Delta B|\psi\rangle = B|\psi\rangle - \langle B\rangle I|\psi\rangle$$

Look at the geometry.

Uncertainty Principle



$$(\Delta A)^2(\Delta B)^2 \geq \left(\langle \Delta A | \Delta B \rangle\right)^2$$

The operators A and B are working in two different bases. They operate on the same state, $|\psi\rangle$, but in different basis states. Position and momentum, say. Imagine the state $B|\psi\rangle$ is projecting out of the plane of the paper, in that different basis, rotating its uncertainty $\Delta B|\psi\rangle$ along with it. We're interested in calculating $(\Delta A)(\Delta B)$. We'll use Schwarz. In preview, though, what we'll be interested in is that projection of ΔA onto ΔB. That tells all.

We're out to prove

$$\Delta A \Delta B \geq \left| \langle \psi | \frac{1}{2i} [A, B] | \psi \rangle \right|$$

We'll use the Schwarz form

$$\Delta A^2 \Delta B^2 \geq \left(\langle \psi | \frac{1}{2i} [A, B] | \psi \rangle \right)^2$$

Idea is to convert that rhs to something Schwarz. Here we go.

Let

$$A|\psi\rangle - \langle A\rangle I|\psi\rangle \equiv f|\psi\rangle$$

and

$$B|\psi\rangle - \langle B\rangle I|\psi\rangle \equiv g|\psi\rangle$$

Schwarz gives

$$\langle f|f\rangle\langle g|g\rangle \geq (\langle f|g\rangle)^2$$

We focus on that rhs. First some basic, straight off the Argand plane complex geometry.

$$(\langle f|g\rangle)^2 = Im(\langle f|g\rangle)^2 + Re(\langle f|g\rangle)^2$$

Left side is complex. Imaginary and real. Pythagoras separates them. Let's see where that leads.

$$Im(\langle f|g\rangle)^2 = \frac{1}{2i}(\langle f|g\rangle - \langle g|f\rangle)$$

Again, that's just manipulation of the complex. $\frac{1}{2i}(a+bi)-(a-bi)=b$, the imaginary component of the complex z .

Similarly,

$$Re(\langle f|g\rangle)^2 = \frac{1}{2}(\langle f|g\rangle + \langle g|f\rangle)$$

Now it turns out the real component usually doesn't affect the Schwarz inequality. What matters is the imaginary component. That's because the real term is always positive and always, well, real. It's that differential in the imaginary component that's tracking uncertainty. It's the imaginary component that takes us out of the real plane and into the variance due to measurement in different bases. A measurement with meter sticks vs. a measurement with stopwatches, say. What's the variance in outcome when you have to consider both the error in your meter stick and the error in the stopwatch. Measure a meter a bit too long and a time too short, then measure a meter a bit too short and a time too long and you've multiplied your variance. That's what's included out there in the angle theta between the uncertainty vectors. At least I think . . . Anyway, we'll take Zwiebach at his word and focus on that imaginary component. Time to plug back in the operators.

$$Im(\langle f|g\rangle)^2 = \frac{1}{2i}(\langle\psi|\,(A - \langle A\rangle)(B - \langle B\rangle)|\psi\rangle) - (\langle\psi|\,(B - \langle B\rangle)(A - \langle A\rangle)|\psi\rangle)$$

Do the algebra.

$$\frac{1}{2i}(\langle\psi|\,(A - \langle A\rangle)(B - \langle B\rangle)|\psi\rangle) - (\langle\psi|\,(B - \langle B\rangle)(A - \langle A\rangle)|\psi\rangle)$$

$$= \frac{1}{2i}(\langle\psi|\,(AB - A\langle B\rangle - \langle A\rangle B + \langle A\rangle\langle B\rangle)|\psi\rangle) - (\langle\psi|\,(BA - B\langle A\rangle - \langle B\rangle A + \langle B\rangle\langle A\rangle)|\psi\rangle)$$

Good. Now take a look. Those are expectation values with mixed terms, scalars (other expectation values), and operators for which the bracketing psi's will calculate expectation values. For example pull out that first $A\langle B\rangle$ term. We're working in linear vector spaces after all. We can separate terms.

$$\langle\psi|\,(-A\langle B\rangle)|\psi\rangle = -\langle B\rangle^*\langle\psi|\,(A)|\psi\rangle = -\langle B\rangle^*\langle A\rangle$$

Product of two expectation values. Itself a number.

Well, if you simplify all those manipulations inside the original brackets,

$$Im(\langle f|g\rangle)^2 = \frac{1}{2i}(\langle\psi|AB - \langle A\rangle\langle B\rangle\,|\psi\rangle) - (\langle\psi|BA - \langle B\rangle\langle A\rangle\,|\psi\rangle)$$

By the rules, collect the terms in the brackets.

$$Im(\langle f|g\rangle)^2 = \frac{1}{2i}(\langle\psi|AB - \langle A\rangle\langle B\rangle - BA + \langle B\rangle\langle A\rangle\,|\psi\rangle) = \frac{1}{2i}(\langle\psi|\,[A,B]\,|\psi\rangle)$$

As promised! Pretty slick! Look how that commutator is telling us the uncertainty. It's the variance between measurement in different bases.

Next up: derivation of the Energy-time uncertainty relation. Now we've got the general formula; all we have to do is plug in likely operators.

Here's the idea. Start with the time-dependent SE and a dummy operator, Q, that is time independent.

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$$

Bracket the commutator of H with Q .

$$\langle\psi|\,[H,Q]\,|\psi\rangle = \langle\psi|\,HQ - QH\,|\psi\rangle = \langle\psi|\,HQ\,|\psi\rangle - \langle\psi|\,QH\,|\psi\rangle$$

$$= \langle H\psi|\,Q\,|\psi\rangle - \langle\psi|\,Q\,|H\psi\rangle = -i\hbar\frac{\partial}{\partial t}\langle\psi|\,Q\,|\psi\rangle - i\hbar\frac{\partial}{\partial t}\langle\psi|\,Q\,|\psi\rangle = -2i\hbar\frac{\partial}{\partial t}\langle Q\rangle$$

Let’s check that. Work backward.

$$\frac{\partial}{\partial t}\langle Q\rangle = \frac{\partial}{\partial t}\langle\psi|\,Q\,|\psi\rangle = \left\langle\frac{\partial}{\partial t}\psi\right|\,Q\psi\rangle + \left\langle\psi|\,Q\frac{\partial}{\partial t}\psi\right\rangle = \frac{1}{i\hbar}(\langle H\psi|\,Q\,|\psi\rangle - \langle\psi|\,Q\,|H\psi\rangle)$$

$$= \frac{1}{i\hbar}(\langle\psi|\,HQ\,|\psi\rangle - \langle\psi|\,QH\,|\psi\rangle) = \frac{1}{i\hbar}(\langle\psi|\,HQ - QH\,|\psi\rangle)$$

$$= \frac{1}{i\hbar}\langle\psi|\,[H,Q]\,|\psi\rangle$$

Looks promising, but I’ve lost some factors in there somewhere.

Note that $\langle Q\rangle$ is time dependent. Plug back in to Schwarz.

$$(\Delta H)^2(\Delta Q)^2 \geq \left(\langle\psi|\,\frac{1}{2i}[H,Q]\,|\psi\rangle\right)^2 = \left(-\hbar\frac{\partial}{\partial t}\langle Q\rangle\right)^2$$

$$\rightarrow\ \Delta H\Delta Q = \left|-\hbar\frac{\partial}{\partial t}\langle Q\rangle\right|$$

Now identify a time interval

$$\Delta t = \frac{\Delta Q}{\frac{\partial\langle Q\rangle}{\partial t}}$$

the time interval required for the operator Q to change by the amount ΔQ .

Done! Well, factor of two or so. $\Delta H\Delta t = \hbar$. Identify ΔH with its eigenvalue and we’ve got it.

$$\Delta E\Delta t = \hbar$$

Good enough.

Lecture 11: Uncertainty and the spectral theorem

Professor Z. checks out the position-momentum uncertainty with an example,

$$H = \frac{p^2}{2m} + x^4$$

First convert to operators, then carry on. Gives pretty good prediction for the experimental values. (Not sure where he gets those experimental data.) See the notes.

On to diagonalization. Pretty straightforward. If an operator has a complete set of eigenvectors on a vector space, then you can diagonalize the operator, eigenvalues along the diagonal with corresponding eigenvectors. Better yet, you can find an orthonormal diagonalization: eigenvalues along the diagonal and orthonormal eigenvectors.

Here are the necessary transformations. Given

$$Tv = \lambda v$$

with operator $T \in \mathcal{L}(V)$ and $v = \{v_i\}$ a set of basis vectors on V. Then there exists an operator A such that $Av_i = u_i$, $u = \{v_i\}$ another set of basis vectors on V and

$$Tu = \lambda u$$

T diagonal.

That's possible because you can choose v orthonormal so that

$$u_i = A_{ik} v_k = \begin{bmatrix} A_{1k} \\ A_{2k} \\ \vdots \\ A_{nk} \end{bmatrix}$$

because

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

etc.

And even better yet, you start with an orthonormal v and choose a unitary operator A such that u also is an orthonormal basis. That gives T as simple as possible: a diagonal matrix, all the eigenvalues along the diagonal, with orthonormal eigenvectors.

How find that T ? Assume v orthonormal and A unitary as per above.

$$Tv = \lambda v \ \rightarrow \ TAv = \lambda Av \ \rightarrow \ A'TAv = A'\lambda Av = \lambda v$$

$$A'Tu = \lambda v$$

There's your diagonalized matrix, $A'T$.

I've skipped over some subscripts. Those lambda's, of course, are different in the different bases. But onward.

Main idea in this lecture, and one of the key ideas for all Heisenberg-Dirac QM is the spectral theorem. If an operator T in vector space V has eigenvectors that form an orthonormal basis in V, then T must be a normal operator. By definition a normal operator has the property that

$$[T^{\dagger}, T] = 0$$

Normal operators include Hermitian and unitary operators. They're the ones we need for quantum mechanics.

The rest of the lecture involves proofs for spectral theorem and various properties of diagonalization. I guess the practical takeaway for all this is if you can diagonalize an operator, then you immediately see it's spectrum; all the eigenvalues sit there along the diagonal. And all the eigenvectors are just 1's in the proper slot on the unit vectors.

Oh yes. Other idea here is figuring out under what conditions you can simultaneously diagonalize two operators. Key requirement is that the two operators must have bases of the same dimension in the same vector space. Then you can relate one to the other by converting the bases. Diagonalize one and you know how to diagonalize the other by change of basis.

Complications enter when you have a degenerate operator. Same energy, say, for several different eigenstates. You want to be able to find out what makes those states different if it's not the energy. That's the whole point of the simultaneous diagonalization of different operators. Represent one operator in terms of the other and you figure out what distinguishes them.

The matrix math is interesting. Turns out that the diagonalization of a degenerate operator gives you a block diagonal matrix. Each block represents the subspace of eigen's with the same energy. That's the conceptually important point. Degenerate eigenstates mean you have several vectors (states), a vector subspace, all with the same energy. Each energy is a subspace of the larger vector space. Add up all those subspaces, a direct sum, and you get the entire vector space. Anyway, once you've figured out the blocks you can diagonalize the individual blocks with a new set of matrix transformations inherent to that subspace. Easy peasy. Well, there's a whole lot of matrix calculations going on. But certainly do-able.

Lecture 12: Harmonic oscillator and quantum dynamics

Harmonic oscillator. Again? For heaven's sake, why?

Well, we've got new tools and we can use them to pry further under the hood. Normal operators and the spectral theorem provide new understanding. Nuts and bolts. We get to see more of the blueprints, how quantum mechanics is put together.

Start with the usual

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

Factor. Purpose here is to produce a form $H \sim V^\dagger V$. Why? So we can apply the spectral theorem. Hang on to your hats.

$$H = \frac{1}{2} m\omega^2 \left(\hat{x} - \frac{\hat{p}}{m\omega}\right)\left(\hat{x} + \frac{\hat{p}}{m\omega}\right) + \frac{\hbar}{2}\omega$$

Last term there, remember, results from the cross terms and their commutation relations.

Okay. So what? Well, those parentheses are normal operators! Let

$$\left(\hat{x} - \frac{\hat{p}}{m\omega}\right) = V^\dagger \text{ and } \left(\hat{x} + \frac{\hat{p}}{m\omega}\right) = V$$

Normal operators. Check it out. Most obvious is that their product gives a Hermitian operator, H. So their (orthonormal) eigenvectors provide a basis for their complex (Hilbert) vector space. So they are diagonalizable, and their eigenvalues sit on the diagonals. We just have to read off the diagonals to find, e.g., the spectrum of the harmonic oscillator! Can't get any spiffier than that!

On to the nit and grit and an interesting aha! Define

$$\hat{a} \equiv \sqrt{\frac{m\omega}{2\hbar}}V \text{ and } \hat{a}^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}V^\dagger$$

Then

$$H = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

Tidy! Even tidier a number operator $N \equiv \hat{a}^\dagger\hat{a}$. It's just going to tell us which energy state we're in. It's an integer. With that

$$H = \hbar\omega\left(N + \frac{1}{2}\right)$$

Some consequences of importance:

$$[\hat{a}^\dagger, \hat{a}] = -1 \ \ and \ \ [\hat{a}, \hat{a}^\dagger] = 1$$

Check it out. With that, now here's cool and illustrative. Remember working on the Energy-time uncertainty we found that

$$\frac{\partial}{\partial t}\langle Q\rangle = \frac{i}{\hbar}[H, Q]$$

Where Q is any general operator. Flip the other way, the commutator of the Hamiltonian on an operator is the time derivative of that operator with that quantum factor coefficient $\frac{i}{\hbar}$. Well, looky here.

$$\frac{i}{\hbar}[H, \hat{a}] = \frac{\partial}{\partial t}\hat{a} = \frac{i}{\hbar}(-\hbar\omega\hat{a}) = -i\omega\hat{a}$$

Check that out. The sequence of equations comes from

$$[H, \hat{a}] = -\hbar\omega\hat{a}$$

You can show that with the essential properties of the ladder operators $\hat{a}, \hat{a}^\dagger$ and their commutation relations.

Anyway, where have we seen that?

$$\frac{\partial}{\partial t}\hat{a} = -i\omega\hat{a}$$

$$\hat{a} = e^{-i\omega t}$$

Plucked right out of

$$\psi(t) = e^{-i\omega t}\psi(0) = e^{-iEt/\hbar}\,\psi(0)$$

It's the phase operator that tells us how the wavefunction (state) evolves over time! Trumpets! Fanfare! $\hat{a}$ is the operator that gives a state an energy kick up to the next energy state on the ladder.

There's a connection between state space and wave mechanics. Surprise?

Whew! I'm bogged down in the harmonic oscillator. Bogged down with the abstractions. What are the number operators? What's $|E\rangle$? What's it mean $|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^{\dagger})^{n}|0\rangle$? What's the relation of $\psi(x)$ to all the brackets? Maybe back to the big picture to figure out all the parts.

Take a look at that parabolic quantum well.

figure potential well

There's the potential, built into the geometry of the well. There are the number operators, counting rungs up the ladder of the wavefunctions. There's the lowest energy level, down at the bottom, for which you can fit any wave at all. There's the momentum built into the wavenumber at each energy level.

Focus on them counting operators. What information do they contain? Knowing the energy level (number) should provide enough information to fill in all the details. x , the only variable, is determined by the parabolic geometry. So wavefunction, fitting waves into the parabola, is determined. So momentum is also determined (by the $\frac{\partial}{\partial x}$ operator). So all you need to know is the energy level (number operator) and path backward to the wavefunction.

Good! See if we can make it work!

We've got a number state $|E\rangle$. What's the corresponding wavefunction?

Well, figure out the expectations value $\langle E|E\rangle$. No help there, just numbers. Energy levels. But here's what we can do. We know that $\left\langle \hat{a}^{\dagger}E\middle|\hat{a}^{\dagger}E\right\rangle = (N+1)\hbar\omega\,\langle E|E\rangle$. If we just had a bottom line, the lowest energy level, we could bootstrap up to all the others. Well, we've got that.

$$E_0 = \left(N_0 + \frac{1}{2}\right)\hbar\omega$$

$N_0 = 0$ by definition at the lowest energy state. So the ground state energy is $\frac{1}{2}\hbar\omega$ as per we-already-know-that.

Okay. Now find the wavefunction at that lowest state from

$$\hat{a}|\psi_0\rangle = \sqrt{\frac{m\omega}{2\hbar}}\left\langle\psi_0\middle|\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)\middle|\psi_0\right\rangle = 0$$

because the lowering operator acting on the lowest energy state gives you no state at all.

Solve the differential equation and you're set. You've got the ground level energy and you can construct the ladder up from there.

Here's another perspective from R. Shankar's text. Shankar dives into the weeds, but he's careful to emphasize the essentials. Here are the essentials.

$\hat{a}$ and $\hat{a}^{\dagger}$ as defined from the Hamiltonian. From that, note $[\hat{a},\hat{a}^{\dagger}] = 1$, $\left[\hat{H},\hat{a}^{\dagger}\right] = \hat{a}^{\dagger}$ and $\left[\hat{H},\hat{a}\,\right] = -\hat{a}$. Them's just from the algebra of those operators. You can work it out. Note that the coefficients of the raising and lowering operators convert to units of energy. Of course.

And don't forget, most important of the operators for building the ladder:

$$H = \left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right)\hbar\omega = \left(n + \frac{1}{2}\right)\hbar\omega$$

where n is the number operator. It just records which rung on the energy ladder we're at. Makes sense, $\hat{a}^{\dagger}a \equiv n$. Operate on an energy eigenstate with a takes you down a rung; then operate with $\hat{a}^\dagger$ and you're back where you started, on the same energy level. Same number rung. And makes sense for the energy. Energy of the ground state $|0\rangle$ is

$$\left(n+\frac{1}{2}\right)\hbar\omega = \left(0+\frac{1}{2}\right)\hbar\omega = \frac{1}{2}\hbar\omega$$

Energy of the second energy level

$$\left(1+\frac{1}{2}\right)\hbar\omega = \frac{3}{2}\hbar\omega$$

and so on.

Now rescale into quantum units of $\hbar\omega$. Operator $\widehat{H} \equiv \frac{H}{\hbar\omega}$ and eigenvalues $\varepsilon \equiv \frac{E}{\hbar\omega}$. Numbers. These will scale the energy ladder of the harmonic oscillator into unit steps of $\hbar\omega$.

Number operators. So we can just label the rungs of the energy ladder by their numbers. Call them $|n\rangle$. Energy states in the harmonic oscillator labeled by which rung of the ladder. That's all we need to know to get the rest of the information about the state, e.g. it's particular energy. That's all we need to know to figure out the spectrum.

At which we should take a moment to review Shankar's take-aways from this study of the sho in a parabolic potential.

1. Energy is quantized.
2. There are unit steps $\hbar\omega$ between energy levels.
3. Ground state energy is $\frac{\hbar\omega}{2}$, not zero. (We'll get there shortly.)
4. Wavefunctions leak out beyond the potential well, as per Adams' discussion of exponential decay at the boundaries, etc.
5. Wavefunctions are either symmetric or antisymmetric in the potential well. i.e. cosine-like wavepackets or sine-like.
6. Because of 5. above, the probability distribution for locating a particle does not follow the classical predictions. (Compare graphs of $|n_0\rangle$ and $|n_1\rangle$. Particle most probably at the center of the well in the ground state but probability zero to be found at the center in $|n_1\rangle$.

Back to derivation of the energy ladder and, coolest of cool, the matrix representation of energy eigenstates in Hilbert space.

Consider the Hamilton operator acting on a number state.

$$\widehat{H}a|\varepsilon\rangle = (\epsilon - 1)a|\varepsilon\rangle$$

Work it out.

$$\widehat{H}a|\varepsilon\rangle = \left(a\widehat{H} + \left[\widehat{H}, a\right]\right)|\varepsilon\rangle = (a\varepsilon - a)|\varepsilon\rangle = (\varepsilon - 1)a|\varepsilon\rangle$$

as advertised. We've substituted the energy eigenvalue ε for the operator in the last steps.

Take a close look. What that's telling us is that $(\varepsilon - 1)$ is the eigenvalue for the operator $\widehat{H}$ acting on the original state $a|\varepsilon\rangle$. Since the number operators identify the states, then $a|\varepsilon\rangle$ itself must be the same as the state $|\varepsilon - 1\rangle$ up to a phase coefficient. Get that? Since the number operator are telling us everything we need to know about the energy states, if an operator transforms original state $a|\varepsilon\rangle$ into state $(\varepsilon - 1)a|\varepsilon\rangle$ then $a|\varepsilon\rangle$ must equal $|\varepsilon - 1\rangle$ up to a phase factor. $\widehat{H}a|\varepsilon\rangle = (\varepsilon - 1)a|\varepsilon\rangle = \widehat{H}|\varepsilon - 1\rangle = (\varepsilon - 1)|\varepsilon - 1\rangle$ up to a phase factor.

$$a|\varepsilon\rangle = C_n|\varepsilon - 1\rangle$$

Similarly we can show that

$$\widehat{H}\hat{a}^{\dagger}|\varepsilon\rangle = (\epsilon + 1)\hat{a}^{\dagger}|\varepsilon\rangle = (\epsilon + 1)C_{n+1}|\varepsilon + 1\rangle$$

Lowering and raising operators. Down and up the energy ladder. Up is okay. We can go forever up. Down must have a floor. A minimum ground state energy. What is it? Here's one demonstration the coolness of Shankar's approach.

At the ground state

$$a|\varepsilon_0\rangle = 0$$

Note the zero here is not a state, not an eigenvalue. It's a representation of nada. Such a condition, a state below the ground state, does not exist.

Well, if you act on a non-existing state with the raising operator, you still get nada.

$$\hat{a}^{\dagger}a|\varepsilon_0\rangle = 0$$

Now apply the operator relations.

$$\hat{a}^{\dagger}a|\varepsilon_0\rangle = \left(\hat{H} - \frac{1}{2}\right)|\varepsilon_0\rangle = \left(\varepsilon_0 - \frac{1}{2}\right)|\varepsilon_0\rangle = 0$$

So

$$\varepsilon_0 = \frac{1}{2}$$

or, translated back into units of $\hbar\omega$

$$E_0 = \frac{\hbar\omega}{2}$$

Ground state energy. $\frac{\hbar\omega}{2}$. That's how we figure it out. Back and forth between number operators and the physical Hamiltonian.

Now let's go after those coefficients to the ladder operators. Here things get a little confusing with labels. Shankar (and Zwiebach, also) seem to be using the number labels and the energy level labels interchangeably. No reason not to, I guess. They both represent which step we are on the (quantized) energy spectrum. Anyway, using the number labels now:

Given

$$a|n\rangle = C_n|n-1\rangle$$

it follows

$$\langle n|\hat{a}^{\dagger}a|n\rangle = \langle n-1|{C_n}^{\dagger}C_n|n-1\rangle$$

By previous definition, $\hat{a}^{\dagger}a \equiv n$, the number operator. Rewrite.

$$\langle n|n|n\rangle = \langle n-1|{C_n}^{\dagger}C_n|n-1\rangle$$

So

$${C_n}^{\dagger}C_n = {C_n}^2 = n$$

$$C_n = \sqrt{n}$$

Coefficient for the eigenstate generated by the lowering operator is $\sqrt{n}$, the square root of the energy level of the original state. Similarly, coefficient for the eigenstate after the raising operator is $\sqrt{n+1}$.

$$\hat{a}^{\dagger}|n\rangle = \sqrt{n+1}\ \ |n+1\rangle$$

And now – drum roll! – we can put it all together in operator matrix and normalized eigenstate vector form. Take a look.

$$a = \begin{bmatrix} 0 & 1 & 0 & \cdots & & & \\ 0 & 0 & \sqrt{2} & & & & \\ 0 & 0 & 0 & \sqrt{3} & & & \\ \vdots & & & 0 & \sqrt{4} & & \\ & & & & 0 & \ddots & \\ & & & & & 0 & \end{bmatrix}$$

$$a^{\dagger} = \begin{bmatrix} 0 & 0 & 0 & \cdots & & & \\ 1 & 0 & 0 & & & & \\ 0 & \sqrt{2} & 0 & & & & \\ \vdots & & \sqrt{3} & 0 & & & \\ & & & \sqrt{4} & 0 & & \\ & & & & \ddots & 0 & \end{bmatrix}$$

$$E = \hbar\omega \begin{bmatrix} \frac{1}{2} & 0 & & \cdots & & \\ 0 & \frac{3}{2} & & & & \\ & & \frac{5}{2} & & & \\ \vdots & & & \frac{7}{2} & & \\ & & & & \frac{9}{2} & \\ & & & & & \ddots \end{bmatrix}$$

So when you raise the $n = \varepsilon = 3$ eigenstate, for example

$$\hat{H}a^{\dagger}|3\rangle = \hbar\omega \begin{bmatrix} \frac{1}{2} & 0 & & \cdots & & \\ 0 & \frac{3}{2} & & & & \\ & & \frac{5}{2} & & & \\ \vdots & & & \frac{7}{2} & & \\ & & & & \frac{9}{2} & \\ & & & & & \ddots \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \cdots & & \\ 1 & 0 & 0 & & & \\ 0 & \sqrt{2} & 0 & & & \\ \vdots & & \sqrt{3} & 0 & & \\ & & & \sqrt{4} & 0 & \\ & & & & \ddots & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

$$= \hbar\omega \begin{bmatrix} \frac{1}{2} & 0 & & \cdots & & \\ 0 & \frac{3}{2} & & & & \\ & & \frac{5}{2} & & & \\ \vdots & & & \frac{7}{2} & & \\ & & & & \frac{9}{2} & \\ & & & & & \ddots \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sqrt{3} \\ 0 \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sqrt{3}\left(\frac{7}{2}\right) \\ 0 \\ \vdots \end{bmatrix} \hbar\omega$$

Pretty slick! Take a close look, though, at how those row and column indices are working to raise from energy level 3 to 4.

But that's not quite the whole story. We've got to construct the spectrum from the ground state up, rung by rung up the energy ladder. And we have to normalize. Start with orthonormal eigenstates and you have to maintain orthonormal states. Not too bad, really. Here's the grand finale.

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^{\dagger})^{n}|0\rangle$$

Start with the ground state. Operate n times with the raising operator, as per the example above. Divide each time by the coefficient to maintain unit norm. Ta da! You've got the spectrum of the quantum harmonic oscillator!

Best introduction to next, maybe, is email I sent to Prof. Zwiebach re: his derivation of the Schrodinger equation from the time operator, U :

I am working through your MIT OCW Physics 8.05. In Lecture 12, Dynamics, you derive the Schrodinger equation from the unitary time operator. It struck me that the result is (or sure looks to me like) a continuity equation, where the Hamiltonian operator is a 'current' and the norm of the state vector is a conserved 'charge.'

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \frac{\partial U_0^t}{\partial t} {U_0^t}^\dagger |\psi(t)\rangle$$

A continuity relation seems to make sense, a la Noether, with regard to conservation of energy. Anyway, I was just curious if this notion has any merit. On superficial review of the literature, I find SE as continuity of probability, but I don't see any reference to this way of thinking about continuity of the time-dependent SE.

I'm an old geezer, retired high school teacher, trying to figure out quantum gravity. I've been brushing up on QM. I've sure enjoyed your video lectures, and I've learned a whole lot. Thanks very much for making these ideas accessible to the rest of us.

Here's the full monty. Start with a unitary time operator. It rotates the (normalized) state vector around the Bloch sphere in Hilbert space time step by step.

figure Bloch sphere unitary operator.

Let U_0^t represent the operator that takes the state $|\psi(0)\rangle$ to $|\psi(t)\rangle$.

$$U_0^t\,|\psi(0)\rangle = |\psi(t)\rangle$$

The unitary time operator is unique. It evolves any state to its next time increment. And by definition it's reversible

$$U_0^0 = I$$

and

$$U_t^0 U_0^t = I$$

so

$$U_0^t = {U_t^0}^\dagger$$

I'm using the indices here for general purposes. They could be any $t1$ and $t2$.

Also

$$U_{t2}^{t3}U_{t1}^{t2} = U_{t1}^{t3}$$

With that, we're all set to derive the time-dependent Schrodinger equation. Start with

$$\frac{\partial}{\partial t}|\psi(t)\rangle = \frac{\partial}{\partial t}(U_0^t\,|\psi(0)\rangle)$$

The only time dependence on the rhs is in the unitary operator, so

$$\frac{\partial}{\partial t}|\psi(t)\rangle = \frac{\partial U_0^t}{\partial t}|\psi(0)\rangle$$

We want the same eigenstate, $|\psi(t)\rangle$ on both sides.

$$\frac{\partial}{\partial t}|\psi(t)\rangle = \frac{\partial U_0^t}{\partial t}\,U_t^0|\psi(t)\rangle = \frac{\partial U_0^t}{\partial t}\,{U_0^t}^\dagger|\psi(t)\rangle$$

Call that operator on the rhs Λ .

$$\Lambda \equiv \frac{\partial U_0^t}{\partial t}\,{U_0^t}^\dagger$$

and

$$\Lambda^\dagger \equiv U_0^t\frac{\partial {U_0^t}^\dagger}{\partial t}$$

Note that the lambda operators are currents! Just like

$$\frac{\partial\,\psi(x)}{\partial t}\,\psi^*(x)$$

is the probability current.

Claim is that the lambda operators anti-commute.

$$\Lambda + \Lambda^\dagger = 0$$

Easy to show:

$$\frac{\partial}{\partial t}\left(U_0^t{U_0^t}^\dagger\right) = \frac{\partial}{\partial t}I = \frac{\partial U_0^t}{\partial t}\,{U_0^t}^\dagger + U_0^t\frac{\partial {U_0^t}^\dagger}{\partial t} = \Lambda + \Lambda^\dagger = 0$$

We can convert the lambdas to commuting operators by multiplying a factor $i\hbar$.
Multiply the whole shebang. Then we're set.

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = i\hbar\frac{\partial U_0^t}{\partial t}\,{U_0^t}^\dagger|\psi(t)\rangle = i\hbar\,\Lambda|\psi(t)\rangle$$

Now $i\hbar\,\Lambda$ is a unitary, time-step Hermitian operator. What's in a name? Just call it $\widehat{H}$.

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \widehat{H}|\psi(t)\rangle$$

Schrodinger! And cool thing is, by all appearances it's a continuity equation. Λ hence $\widehat{H}$ is a current. It's a current in time. It's the flow of time.

$$\widehat{H} = i\hbar\frac{\partial U_0^t}{\partial t}\,{U_0^t}^\dagger$$

That derivative-times-operator is a current just like the probability density current $J = \frac{\partial\psi(x)}{\partial x}\,\psi(x)^*$. The conserved 'charge' is the norm of the state vector.

$$\langle\psi(t)|\,i\hbar\frac{\partial}{\partial t}\,|\psi(t)\rangle - \widehat{H} = 0$$

Conservation of energy! A charge. A Noether current. Symmetry. It's all right there. Energy is conserved.

Well, hardly surprising. We've seen that from the commutation relations and a bunch of other things before. Still, pretty cool that it appears in Schrodinger. Or that you can derive Schrodinger from Noether.

<u>Lecture 13</u>: Dynamics (cont'd) and the Heisenberg operator

What's next is to understand the unitary time evolution operator in terms of the Hamilton operator. Calculation generally starts with the Hamiltonian. That's the physics. From that we want to figure out U .

Straightforward. Return to Schrodinger.

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}(U_0^t\,|\psi(0)\rangle) = \widehat{H}|\psi(t)\rangle = \widehat{H}\,(U_0^t\,|\psi(0)\rangle)$$

Since the only time-dependence is in the $U_0^t{}'s$ we can write

$$i\hbar\frac{\partial}{\partial t}U_0^t = \widehat{H}\,U_0^t$$

and solve!

$$U_0^t = e^{-iHt/\hbar}$$

Similarly for a 'slightly' time-dependent Hamiltonian.

$$U_0^t = e^{-iH(t_1 - t_0)/\hbar}$$

Sure looks familiar. Then there's the whole-shebang Hamiltonian-time-dependent:

$$U_0^t = exp\left(-\frac{i}{\hbar}\int_0^t H(t)\, dt\right)$$

That follows from Taylor series and a time ordering operator. See Prof. Z's notes for details. Anyway, makes sense just by the looks of it. Time operator increments step by step over time driven by the Hamiltonian.

Onward to Heisenberg operators.

$$A_H \equiv U_0^{t\dagger} A_S U_0^t$$

where the H and the S subscripts refer to Heisenberg and Schrodinger. Schrodinger operators are all the usual, $\hat{x}, \hat{p}, \widehat{H},$ etc. All the dynamics now is included in the Heisenberg operator.

From that definition

$$\langle\psi(t)|A_s|\psi(t)\rangle = \left\langle\psi(0)\middle|U_0^{t\dagger} A_S U_0^t\middle|\psi(0)\right\rangle = \langle\psi(0)|A_H|\psi(0)\rangle$$

That's handy. The Heisenberg operator allows us to choose an initial state to study the dynamics, and we can stick with that state through our calculations.

This definition gives a bunch of handy relations, all familiar from the usual Schrodinger operators.

$$C_S = A_S B_S \;\rightarrow\; C_H = U_0^{t\dagger} A_S U_0^t U_0^{t\dagger} B_S U_0^t = U_0^{t\dagger} A_S I B_S U_0^t = U_0^{t\dagger} A_S B_S U_0^t = C_H$$

and

$$[A_H, B_H] = \left[U_0^{t\dagger} A_S U_0^t, U_0^{t\dagger} B_S U_0^t\right] = \left(U_0^{t\dagger} A_S U_0^t\right)\left(U_0^{t\dagger} B_S U_0^t\right) - \left(U_0^{t\dagger} B_S U_0^t\right)\left(U_0^{t\dagger} A_S U_0^t\right)$$

$$= \left(U_0^{t\dagger} A_S B_S U_0^t\right) - \left(U_0^{t\dagger} B_S A_S U_0^t\right) = A_H B_H - B_H A_H = [A_H, B_H]$$

sure enough! General rules for any and all operators. Note that out of laziness I've dropped the operator hat symbols.

From this we can prove anew that $i\hbar\frac{\partial}{\partial t}A_H = [A_H, H_H]$.

Start with Schrodinger.

$$i\hbar\frac{\partial}{\partial t}A_H = i\hbar\frac{\partial}{\partial t}U_0^{t\dagger}A_S U_0^t = i\hbar\left(\frac{\partial U_0^{t\dagger}}{\partial t}A_S U_0^t + U_0^{t\dagger}\frac{\partial A_S}{\partial t}U_0^t + U_0^{t\dagger}A_S\frac{\partial U_0^t}{\partial t}\right)$$

Convert the time derivatives of the unitary operators to their Hamiltonians.

$$= -U_0^{t\dagger}\hat{H}_S A_S\, U_0^t + i\hbar\left(U_0^{t\dagger}\frac{\partial A_S}{\partial t}U_0^t\right) + U_0^{t\dagger}A_S\hat{H}_S\, U_0^t$$

$$= -U_0^{t\dagger}\hat{H}_S A_S\, U_0^t + U_0^{t\dagger}A_S\hat{H}_S\, U_0^t + i\hbar\left(U_0^{t\dagger}\frac{\partial A_S}{\partial t}U_0^t\right)$$

$$= [A_H, H_H] + i\hbar\left(U_0^{t\dagger}\frac{\partial A_S}{\partial t}U_0^t\right)$$

Now if A_S the Schrodinger operator has no time dependence, the last term disappears, and we're left with the commutator relation as proved.

Well, maybe not real surprising, since we've seen it before, but it's interesting to see how those Heisenberg operators work.

Take a look now how the Heisenberg picture simplifies our understanding of the physics. Following comes from Prof. Z as well as Shankar's text. The latter complements Prof. Z's lectures really nicely.

First note the geometric relation between the Schrodinger and Heisenberg pictures. Schrodinger's (vector) states rotate around Hilbert space against fixed coordinates (the eigenstates). Heisenberg says the states are fixed and the coordinates rotate. It's the basis vectors, as represented in the operators, that are changing over time. The physics is the same. Calculations come out the same. Just a different way of looking at the world – and Shankar says there's a whole lot of other models we might build. Lots of room for creative thinking.

figure S vs. H coordinates

On to the Heisenberg simplification. H makes the dynamics look the same as in classical physics. For example, in the quantum harmonic oscillator

$$\frac{\partial\hat{x}}{\partial t} = [\hat{x},\hat{H}] = \left[\hat{x},\left(-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2\right)\right] = \left[\hat{x},\left(-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\right)\right]$$

Whoa! What happened to the second term in the Hamiltonian? Well, it commutes with $\hat{x}$ so disappears in the calculations. Might as well drop it from the get-go.

$$\frac{\partial \hat{x}}{\partial t} = -\frac{i}{\hbar}\frac{1}{2m}[\hat{x},\hat{p}^2] = -\frac{i}{\hbar}\frac{1}{2m}([\hat{x},\hat{p}]\hat{p} + [\hat{x},\hat{p}]\hat{p})$$

where that last step pulled out one of the two $\hat{p}$ operators for each term. You have to calculate the commutator twice; once for each $\hat{p}$. So

$$\frac{\partial \hat{x}}{\partial t} = -\frac{i}{\hbar}\frac{1}{2m}([\hat{x},\hat{p}]\hat{p} + [\hat{x},\hat{p}]\hat{p}) = -\frac{i}{\hbar}\frac{1}{2m}(2i\hbar\hat{p}) = \frac{\hat{p}}{m}$$

Just as in classical mechanics! The Heisenberg operators return the classical equation of motion!

Same for $\frac{\partial \hat{p}}{\partial t}$.

$$\frac{\partial \hat{p}}{\partial t} = [\hat{p},\hat{H}] = \left[\hat{p},\left(-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2\right)\right] = \left[\hat{p},\left(\frac{1}{2}m\omega^2\hat{x}^2\right)\right]$$

Same rationale here for dropping the first term in the Hamiltonian. It commutes with $\hat{p}$.

$$\frac{\partial \hat{p}}{\partial t} = \frac{i}{\hbar}\frac{1}{2}m\omega^2[\hat{p},\hat{x}^2] = \frac{i}{\hbar}\frac{1}{2}m\omega^2([\hat{p},\hat{x}]\hat{x} + [\hat{p},\hat{x}]\hat{x})$$

$$\frac{\partial \hat{p}}{\partial t} = \frac{i}{\hbar}\frac{1}{2}m\omega^2(2i\hbar\hat{x}) = -m\omega^2\hat{x}$$

Shankar's other observation deserves repeat. Back to the definition. Because the Heisenberg operators do all the lifting for time evolution we can solve all the dynamics just based on some initial state, which presumably we can determine. Which is the whole point. We know a state to start with and we want to see how it evolves.

Lecture 14: Coherent states

Coherent states are replicates. Well, sort of. They share the same energy, but differ in other observables.

Take the quantum harmonic oscillator, for example. Lowest eigenstate, the ground state, has energy $\frac{1}{2}\hbar\omega$. But shucks. We could move the apparatus over yonder and the ground state over there is the same $\frac{1}{2}\hbar\omega$ as the ground state here. Coherent.

Prof. Zwiebach starts the lecture with a review of dynamics. Position and momentum operators as functions of time.

$$\hat{x}(t) = \hat{x}(0)cos(\omega t) + \frac{\hat{p}}{m\omega}(0)sin(\omega t)$$

with some coefficients in there I need to look up.

$$\hat{p}(t) = \hat{p}(0)cos(\omega t) - m\omega\hat{x}(0)sin(\omega t)$$

New is the Heisenberg dynamics of the ladder operators.

$$\hat{a}_H = e^{-i\omega t}\hat{a}$$

and

$$\hat{a}_H{}^{\dagger} = e^{i\omega t}\hat{a}^{\dagger}$$

Essential tools of the coherent states are translation operators.

$$T_{x_0} \equiv e^{-i\hat{p}x_0/\hbar}$$

What it does is increment the position by an interval x_0 . Maybe a more consistent symbolic representation would be

$$T_x^{x_0} \equiv e^{-i\hat{p}x_0/\hbar}$$

That is, the translation operator takes the state from position x to position $x + x_0$.

Note the parallel to the unitary time operator. U_0^t takes the state from time 0 , some time we choose to call zero on our stopwatch, to some later time t . U_0^t is moving the state through time. $T_x^{x_0}$ is moving the state through space. Which raises the question, is there some relativistic relation $T_x^{x_0 2} - U_0^{t\,2} = S^2$? The metric?

figure incrementing the state vector

Anyway, back to the standard stuff. Given the symbolic representation, the algebra of the translation operator is pretty clear from the physics.

$$T_{x_0}{}^{\dagger} = T_{-x_0} = \left(T_{x_0}\right)^{-1}$$

so

$$T_{x_0}{}^{\dagger}T_{x_0} = I$$

That is, you've translated to a new position then right back to where you started.

$$T_{x_0}T_{y_0} = T_{x_0+y_0}$$

That is, translation steps are additive.

Note the representation for T_{x_0} acting on the position operator.

$$T_{x_0}{}^{\dagger}\hat{x}\,T_{x_0} = \hat{x} + x_0 I$$

Think about that one. Talking operators here. lhs is an operator. It's rotating a state vector.

$$\left\langle\psi(x)\middle|T_{x_0}{}^{\dagger}\hat{x}\,T_{x_0}\middle|\psi(x)\right\rangle = \langle\psi(x)|\hat{x}\,|\psi(x)\rangle + x_0 I$$

Take a look at the state vectors.

figure translation in Hilbert space

$$T_{x_0}|x\rangle = |x + x_0\rangle$$

Switching between state vectors and the wavefunction

$$|\psi\rangle \rightarrow \psi(x)$$

$$T_{x_0}|\psi\rangle \rightarrow \psi(x - x_0)$$

Note the minus sign. Functions. Usual rule for translating functions across the coordinates. Minus sign if you move the function to the right.

Now we've got the tools, on to coherent states. In the quantum harmonic oscillator

$$|\tilde{x}_0\rangle \equiv T_{x_0}|0\rangle = e^{-ipx/\hbar}|0\rangle$$

That's it. By definition, a coherent state is the ground state of the harmonic oscillator translated is position but still in the ground state. Slide the pendulum a bit to the right. Move the snowboard pipe a tad further around the hill. Energy states are unchanged. Coherent.

Here's a bit of an aside. How do you run a unit analysis quickly so you can understand, for example, the coefficients in the expectation values? It crossed my mind a good start is in the equivalents for the Planck constant.

$$\hbar = Et = px$$

Play around a bit, makes sense.

$$E = p\frac{x}{t}$$

standard units for kinetic energy, momentum times velocity.

Or

$$\frac{E}{x} = \frac{p}{t}$$

Hamilton equations of motion.

Take a look then at the position and momentum equations of motion we've derived in our dynamics.

$$\hat{x}(t) = \hat{x}(0)cos(\omega t) + \frac{\hat{p}}{m\omega}(0)sin(\omega t)$$

That coefficient $\frac{\hbar}{m\omega}\hat{p}(0)$ should give units of x. Let's see. First term on the rhs is fine. Second term needs some reckoning. Units.

$$\frac{1}{m\omega}\hat{p}(0) = \frac{p}{E/_{\omega x^2}} = \frac{p}{\hbar/_{x^2}} = \frac{p}{px/_{x^2}} = x$$

where I simplified at the second step using the harmonic oscillator $V = \frac{1}{2}m\omega^2 x^2$ and, third step, the units $\omega = \frac{1}{t}$.

It works! No great surprise, but maybe it will help keep track of the coefficients.

Back to Zwiebach and the coherent states. A few more key ingredients.

$$\langle \tilde{x}_0 | \tilde{x}_0 \rangle = 1$$

General expectation values:

$$\langle \tilde{x}_0 | \hat{A} | \tilde{x}_0 \rangle = \langle 0 | T_{x_0}{}^{\dagger} \hat{A} T_{x_0} | 0 \rangle$$

So, for example,

$$\langle \tilde{x}_0 | \hat{x} | \tilde{x}_0 \rangle = \langle 0 | (x + x_0) | 0 \rangle = x_0$$

$$\langle \tilde{x}_0 | \hat{p} | \tilde{x}_0 \rangle = 0$$

because it's a stationary state, wavefunction just sitting there. No momentum.

$$\langle \tilde{x}_0 | \hat{H} | \tilde{x}_0 \rangle = \langle 0 | \hat{H} | 0 \rangle + \frac{1}{2} m\omega^2 {x_0}^2 = \frac{1}{2}\hbar\omega + \frac{1}{2} m\omega^2 {x_0}^2$$

Energy in the coherent state is augmented by the potential at the (displaced) position x_0 . Makes sense.

And for future reference:

$$\langle \tilde{x}_0 | \hat{x}^2 | \tilde{x}_0 \rangle = {\tilde{x}_0}^2 + \frac{\hbar}{2m\omega}$$

$$\langle \tilde{x}_0 | \hat{p}^2 | \tilde{x}_0 \rangle = \frac{m\omega\hbar}{2}$$

$$\langle \tilde{x}_0 | \hat{x}\hat{p} + \hat{p}\hat{x} | \tilde{x}_0 \rangle = 0$$

Move on to the dynamics. We'll use the good ol' Heisenberg operators.

$$\langle \tilde{x}_0(t) | \hat{A}_S | \tilde{x}_0(t) \rangle = \langle \tilde{x}_0 | \hat{A}_H | \tilde{x}_0 \rangle$$

Try it out on the position operator. See what happens to the coherent state position over time.

$$\langle \tilde{x}_0 | \hat{x}_H | \tilde{x}_0 \rangle = \left\langle \tilde{x}_0 \middle| \hat{x}(0) cos(\omega t) + \frac{\hat{p}(0)}{m\omega} sin(\omega t) \middle| \tilde{x}_0 \right\rangle = x_0 cos(\omega t)$$

As per the general results above, the momentum disappears. As it should. And looky! It's the good old classical equation! Position oscillates around zero!

<u>Lecture 15</u>: Coherent states and squeezed states

Prof. Zwiebach calculates the general coherent state, which includes the ladder operators, and he explains the squeezed state. The math is complicated, and I won't reproduce it here. Just lazy, I guess, but I'm getting antzy, want to move on back to the frontier. Quantum information and gravity. Time to get moving, finish up the QM review. Take a peek, know where to go back when I need to, and move beyond.

By which I really mean to say I should take the time to get it under my belt, but I'm getting burnt out.

Squeezed states are worth some discussion, though. I've always wondered what they were. Prof. Z. explains them well. Here's the notion.

Take a coherent state in the ground state of a Hamiltonian. It has an uncertainty

$$\Delta x_1 = \sqrt{\frac{\hbar}{2m_1\omega_1}}$$

where the subscripts identify the particular Hamiltonian.

Now zap the system into a new Hamiltonian. Take a look at the new uncertainty.

$$\Delta x_2 = \sqrt{\frac{\hbar}{2m_2\omega_2}} = \sqrt{\frac{m_1\omega_1}{m_2\omega_2}}\sqrt{\frac{\hbar}{2m_1\omega_1}}$$

Just the rules of arithmetic. Now if

$$\gamma \equiv \sqrt{\frac{m_1\omega_1}{m_2\omega_2}} < 1$$

as in if the energy of the second state is higher than the first, then the state (think of it as a Gaussian) has been squeezed. It isn't as wide as it was to start with. It's a sharper peak.

figure squeezed state

Want to build a squeezed state you need a squeezing operator.

$$S(\gamma) = e^{-\gamma/_2(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a})}$$

Note that it is quadratic in the ladder operators, and note the order of those operators in the exponent. Annihilation op's have to be to the right, acting first on the state you're squeezing. Otherwise the whole thing blows up, driving the state upward with successive creation operators. Now define the squeezed vacuum state

$$|0_\gamma\rangle = S(\gamma)|0\rangle$$

And look at the applications. LIGO uses the translation and squeezing operators reduce noise in its detection system. The mirrors oscillate a bit. That smears out the gravity wave signal. Solution: squeeze the detector photons so they're less exposed to mirror fluctuations, and translate them to where they should be if the mirror was absolutely quiet.

$$|\alpha, \gamma\rangle = D(\alpha)S(\gamma)|0\rangle$$

Pretty cool! You can do marvelous things. Perform sharper measurements. Send sharper signals.

Lecture 16: Photon coherent states and two-state systems

Idea here is you can write the Hamiltonian for the electromagnetic field in a way that looks like the harmonic oscillator.

$$E = \frac{1}{2}(p^2 + \omega^2 q^2)$$

E here is the electromagnetic field. It's same as the harmonic oscillator without mass. It's reasonable in units: $[pq] = [\hbar]$. So declare

$$H \equiv \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{q}^2)$$

where

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}}\,(\hat{a} + \hat{a}^\dagger)$$

and

$$\hat{p} = \sqrt{\frac{\hbar\omega}{2}}(\hat{a} - \hat{a}^\dagger)$$

so

$$H = \hbar\omega\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) = \hbar\omega\left(N + \frac{1}{2}\right)$$

and N now is the photon energy. Just like ladder operator stuff in the harmonic oscillator. What's it all mean? Well consider the photon. Quantum of the electromagnetic field. Think of it as a coherent state. It's a mode in the stupendous harmonic oscillator well of the electromagnetic universe. It has an associated momentum and potential energy, oscillating as it is between its potential boundaries. Mass on a spring, but not mass. Just the spring oscillating. That said, we can assign the usual ladder operators to the Hamiltonian just as in the QHO we already built. Same maths.

All that said, we can think of the field itself as an operator.

$$\hat{E} = \varepsilon_0\left(e^{-i\omega t}\hat{a} + e^{-i\omega t}\hat{a}^\dagger\right)sin(kz)$$

with field polarized along the z-axis.

Lectures 17 and 18: Two-state systems, ammonia and NMR

Two-state systems, e.g. spin states and the ammonia molecule: you can capture them neatly with a 2×2 Hamiltonian matrix thusly:

$$H = \begin{bmatrix} g_0 + g_3 & g_1 - g_2 \\ g_1 + g_2 & g_0 - g_3 \end{bmatrix} = g_0 I + g_1\sigma_x + g_2\sigma_y + g_3\sigma_z$$

In this mathematical structure, all the dynamics is in some kind of 'precession,' magnetic moment in a magnetic field as the prime example, of course, but same dynamics also in the ammonia molecule and other two-state systems.

Note the rubric to build models.

1. Find a likely Hamiltonian
2. Find the energy eigenstates and eigenvalues
3. Find the expectation values
4. Find the dynamics, i.e. the time evolution coefficients

Ammonia is really interesting. Two-state system, nitrogen either above or below the plane of hydrogen atoms, so it has an electric dipole. Put it in an electric field and you separate molecules by energy Δ above or below the ground state.

figure ammonia molecule

Eigenstates you can label, as per usual

$$|\uparrow\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } |\downarrow\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Look familiar? Then the Hamiltonian becomes

$$H = \begin{bmatrix} \Delta & \varepsilon_0 E \\ \varepsilon_0 E & -\Delta \end{bmatrix}$$

From there you can calculate dynamics, which (no surprise) includes terms like $exp\left(\frac{i\omega t}{\hbar}\right)$ and $cos(\omega t)$, where ω is the Larmour (precession) frequency. And from those dynamics you can calculate how long it takes, time T , for an up state to flip down. And that, my friend, lets you build masers!

Separate states with a gradient electric field. Send up state into a resonant cavity of just the right length such that the transit time = T . Well, that's just right for the molecule to emit a photon of energy $= 2\Delta$. Photons pile up in the cavity. Let them leak out and you've got a maser. Nobel prize for Townes et al in 1964.

figure maser

And it's all right there in those matrix operators and state vectors and a little bit of math. (Well, quite a bit of math.)

NMR uses the same maths tools. Apparatus has a twist to it, though. Well a rotating magnetic field. Put your target nucleus in a really strong B_z field. Add a rotating B field in the *x*-*y* plane. Nuclear spin precesses around B_z and also around the (rotating) B_x . Effect is to torque the spin axis down into the $x - y$ plane. As it spirals down, it radiates at the frequency of the rotating B_x and you tune the detectors to pick that up. You're seeing mostly the hydrogens in water. Strength of the signal depends on the water concentration and the neighboring molecules. You can get even more information from the damping time and relaxation time; how long does it take to spiral down, and how long to revert to alignment along B_z .

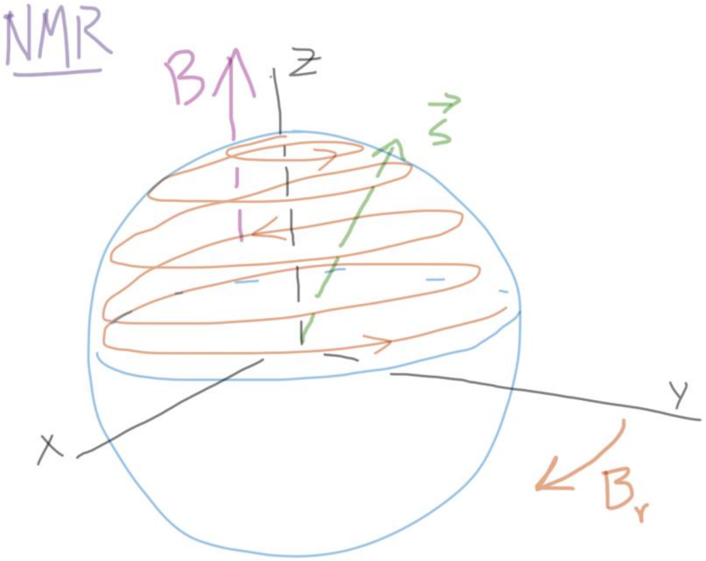


Figure NMR. In a constant external magnetic field B and rotating field in the x-y plane B_r nuclear spin will precess from the z pole down into the x-y plane. The process emits cyclotron radiation as it drops into the plane, and that information is used to construct an image.

Lecture 19: Tensor product and teleportation

I finally get it! Tensor product is not really a multiplication. It's a record-keeping system for multiparticle states.

Main idea is that you can't describe a multiparticle system just by listing the individual properties of all the component particles. It's not enough to know the position and momentum of each individual particle. Those particles are correlated. Their wavefunctions interact. You have to keep track of all those correlations. All that entanglement.

So if V is the Hilbert space of one particle and W the Hilbert space of a second particle, then the Hilbert space of a system with both particles is $V \otimes W$. For example, given two spin-half particles, their vector spaces

$$V = W = \{\,|+\rangle\,, |-\rangle\,\}$$

and

$$V \otimes W = \{\,|+\rangle \otimes |+\rangle\,, |+\rangle \otimes |-\rangle\,, |-\rangle \otimes |+\rangle\,, |-\rangle \otimes |-\rangle\,\}$$

By convention we'll typically drop the $\otimes$ between state vectors.

$$V \otimes W = \{\,|+\rangle|+\rangle\,, |+\rangle|-\rangle\,, |-\rangle|+\rangle\,, |-\rangle|-\rangle\,\}$$

Note that the dimension of the tensor product state is the product of the dimensions of the two component states. All the usual rules of linear algebra apply: scalar coefficients distribute and so do vector states.

$$au \otimes (v \otimes w) = a(u \otimes v) \otimes a(u \otimes w)$$

And with that you can do all kinds of marvelous things. Like for example build a quantum teleportation system. The physical system in the following example uses spin states. Spin operators are unitary, so conserve probability and information. You implement the operators with varying magnetic fields. Them's what goes into the Hamiltonians we call quantum logic gates. Them's what's the physical instantiation of the operators. Magnets.

Here's an illustration of spin operators on Bell states. We'll use them for teleportation. Define the Bell state

$$|\phi_0\rangle \equiv \frac{1}{\sqrt{2}}(|+\rangle|+\rangle\; + |-\rangle|-\rangle\,)$$

Unitary. Normalized. Perfect. Now operate with the spin operators. Note that we have to use augmented operators (e.g. $(I \otimes \sigma_1)$) since we have a two-particle system.

$$|\phi_1\rangle = (I \otimes \sigma_1) \otimes |\phi_0\rangle = \frac{1}{\sqrt{2}}(|+\rangle|-\rangle\; + |-\rangle|+\rangle\,)$$

Think about that. The I in the operator preserves the state of the first particle. σ_1 flips up to down and vice versa, acting on the second particle. Remember the matrix and vector representations. Similarly

$$|\phi_2\rangle = (I \otimes \sigma_2) \otimes |\phi_0\rangle = \frac{i}{\sqrt{2}}(|+\rangle|-\rangle\; - |-\rangle|+\rangle\,)$$

$$|\phi_3\rangle = (I \otimes \sigma_3) \otimes |\phi_0\rangle = \frac{1}{\sqrt{2}}(|+\rangle|+\rangle\; - |-\rangle|-\rangle\,)$$

Work backwards to the paired states. We'll need those, too.

$$|+\rangle|+\rangle = \frac{1}{2}(|\phi_0\rangle + |\phi_3\rangle)$$

$$|+\rangle|-\rangle = \frac{1}{2}(|\phi_1\rangle - i|\phi_2\rangle)$$

$$|-\rangle|+\rangle = \frac{1}{2}(|\phi_1\rangle + i|\phi_2\rangle)$$

$$|-\rangle|-\rangle = \frac{1}{2}(|\phi_0\rangle - |\phi_3\rangle)$$

OK. Teleportation. Alice and Bob share a Bell state

$$|\phi_0\rangle = \frac{1}{\sqrt{2}}(|+\rangle|+\rangle + |-\rangle|-\rangle)$$

Alice grabs the state she wants to teleport to Bob.

$$|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$$

She interacts her states to form a tensor product $|\phi_0\rangle \otimes |\psi\rangle$. The subscripts track spins held by Alice and Bob and, C , the spins to be teleported. The tensor product represents the whole system: spins of all three particles: the entangled pair and the state to be transported.

$$|\phi_0\rangle_{AB} \otimes |\psi\rangle_C = \frac{1}{\sqrt{2}}\alpha(|+\rangle_A|+\rangle_C|+\rangle_B + |+\rangle_A|-\rangle_C|+\rangle_B) \\ + \frac{1}{\sqrt{2}}\beta(|-\rangle_A|+\rangle_C|-\rangle_B + |-\rangle_A|-\rangle_C|-\rangle_B)$$

Well now. We can identify those leading $A \otimes C$ states with Bell bases.

$$|\phi_0\rangle_{AB} \otimes |\psi\rangle_C = \frac{1}{2}\alpha((|\phi_0\rangle + |\phi_3\rangle)_{AC}|+\rangle_B + (|\phi_1\rangle - i|\phi_2\rangle)_{AC}|+\rangle_B) \\ + \frac{1}{2}\beta((|\phi_1\rangle + i|\phi_2\rangle)_{AC}|-\rangle_B + (|\phi_0\rangle - |\phi_3\rangle)_{AC}|-\rangle_B)$$

Regroup that last equation as factors of the Bell bases.

$$|\phi_0\rangle_{AB} \otimes |\psi\rangle_C = \frac{1}{2}|\phi_0\rangle_{AC}(\alpha|+\rangle_B + \beta|-\rangle_B) + \frac{1}{2}|\phi_1\rangle_{AC}(\alpha|-\rangle_B + \beta|+\rangle_B) \\ + \frac{1}{2}i|\phi_2\rangle_{AC}(\alpha|-\rangle_B - \beta|+\rangle_B) + \frac{1}{2}|\phi_3\rangle_{AC}(\alpha|+\rangle_B - \beta|-\rangle_B)$$

Now look at that! Each term on the right is the AC Bell state times the associated spin operator (reflected in the signs; look close) on $|\psi\rangle_B$. Alice has teleported $|\psi\rangle$ to B ! All Bob has to do is operate on his state with the appropriate spin operator. Alice has to send him that information, which operator. Done!

$$|\phi_0\rangle_{AB} \otimes |\psi\rangle_C = \frac{1}{2}\sum_{j=0}^{3} |\phi_j\rangle \otimes \sigma_j \, |\psi\rangle_B$$

Be sure to check Nielsen's quantum circuit for comparison. I think I finally see where that circuit comes from.

On to EPR and the Bell inequality. Zwiebach has a really nice discussion of the argument. I've been confused about what is local realism and other such truck. Turns out it's right there in the two assumptions Einstein insists on:

1. Any measurement reflects a reality of the system. i.e., if you measure a particle has spin up, then that particle most definitely had spin up before the measurement. QM, of course, says that the particle was in a superposition of states before the measurement.
2. Conditions far away cannot affect measurements right here in the lab. QM, on the other hand, says particles can be entangled, i.e. correlated, over vast distances.

Bell's inequality established what's what. It's straightforward in Zwiebach's presentation. Key is that for a spin system the probability measuring spin along some axis at angle theta from the reference axis

$$P = \frac{1}{2} sin^2 \frac{\theta}{2}$$

Check the math. Maybe it's in Adams' notes a while back. But I'm pretty sure I verified this myself.

OK. Here's Professor Z's argument. Consider an experimental apparatus that can measure particle spin along any of three axes. Prepare entangled pairs. The table below lists all the possible entangled states. $+$ and $-$'s are spins along the three axes a, b, c . Columns list the entangled pairs. State labels are arbitrary, just a counting device.

state	particle A	particle B
N_1	$+++$	$---$
N_2	$++-$	$--+$
N_3	$+-+$	$-+-$
N_4	$+--$	$-++$
N_5	$-++$	$+--$
N_6	$-+-$	$+-+$
N_7	$--+$	$++-$
N_8	$---$	$+++$

Figure the classical probabilities, what EPR predicted assuming local realism. Calculations list spin state for particle A followed by state for particle B.

$$a(+)b(+) = N_3 + N_4$$

$$a(+)c(+) = N_2 + N_4$$

$$b(+)c(+) = N_2 + N_6$$

From those equations, it's clear that

$$P[a(+)c(+)] \leq P[a(+)b(+)] + P[b(+)c(+)]$$

Classically. According to EPR. But QM says it ain't so! Suppose the angles between axes are small and the b axis lies between a and c.

figure axes

Those probabilities are

$$\frac{1}{2} sin^2\theta \ ?\leq \ \frac{1}{2} sin^2\frac{\theta}{2} + \frac{1}{2} sin^2\frac{\theta}{2} = sin^2\frac{\theta}{2}$$

And no! Not so! At small angles

$$\frac{1}{2} sin^2\theta \cong \frac{1}{2}\theta^2 \geq \frac{1}{4}\theta^2$$

That's the QM prediction. Alain Aspect and many others have carried out the experiments. QM wins.

<u>Lectures 21-23</u>: Angular momentum

Whole lot of really dense definitions and proofs here. Main point is to develop angular momentum operators, general J that includes all the maths of orbital angular momentum, spin, and everything.

Main points, after a lot of work (which I really need to figure out sometime) is

$$[J_i, J_j] = i\hbar\, \varepsilon_{ijk}\, J_k$$

Quantized. And note that $\hbar$ is units of angular momentum. Like $x \cdot p$ and $E \cdot t$. Think about those relations. And make them all operators. $\hat{x} \cdot \hat{p}$ is obvious; that's the classical angular momentum. $\hat{E} \cdot \hat{t}$ needs some more thinking.

Anyway, after all the maths gymnastics we end up with a Hamiltonian for a spin in a central potential. Like an electron in the electromagnetic potential of its nucleus.

$$H = \frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{\hbar^2}{2mr^2}\left(\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\sin\theta\,\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial^2\phi}\right) + V(r)$$

Note all those accelerations along the radial and angular directions. Seems a quick shorthand to think about it, anyway.

From that Hamiltonian you can calculate that the energy is quantized with quantum numbers j increments of $\frac{1}{2}\hbar$ components along the z-axis and total momentum between $\mp j$. There. I think that about captures it.

The generalized wavefunction, with all the angular and radial terms collected into Y and u operators (via algebra to combine all the messy coefficients)

$$\psi_{Elm} = \frac{u_{El}(r)}{\hat{r}} + Y_{lm}(\theta, \rho)$$

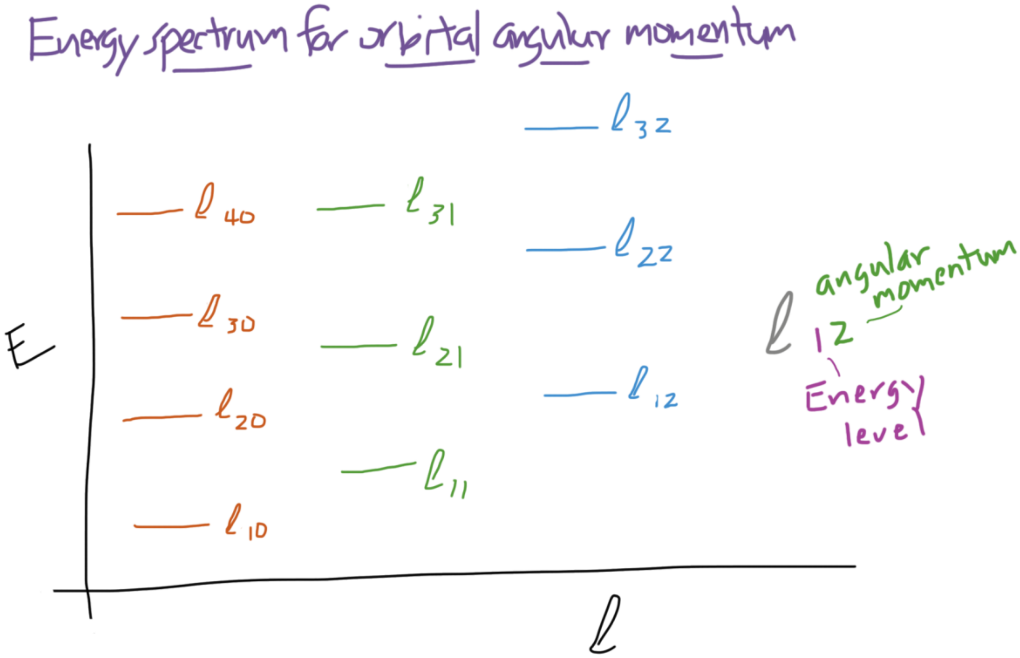
where ρ is a function of radial distance. What that psi is all about: the wavefunction depends on radial distance and spherical angle. There’s just a whole lot of calculation goes into figuring those. From that the Hamiltonian

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}u_E(r) + V_{eff}u_{El} = Eu_{El}$$

where the effective potential, which includes a term for centrifugal force

$$V_{eff} = V(r) - \frac{\hbar^2 l(l+1)}{2mr^2}$$

Take-home from all this – Zwiebach’s words – is summarized in the graph for orbital angular momentum. Note we’re talking orbital L here, not spin, so energy levels are unit quantized and not half-integer. Beyond the l_{i0} (ith energy level with zero angular momentum, as in $s^1, s^2, s^3,$ etc. orbitals) the states are degenerate. So, for example, there are three states at each l_{i1} , one for each of the three l_z components (labeled by m in the algebra). (And note, by the bye, there are $m = 2l + 1$ z-components of angular momentum. See why?)



Take a look here how the spectrum is higgledy-piggledy, not nice and neat like the QHO spectrum below, or the Hydrogen spectrum.

Interesting also is that calculating the wavefunctions on a 2-d surface requires 3-d angular momentum. In fact, the L_{xyz} operators emerge naturally from the algebra. (Don't ask me to demonstrate that right off. Check out the lecture notes. It's there somewhere . . .)

What do those wavefunctions look like? 3-d angular momentum? Well, they're Bessel functions. Okay. So what's a Bessel function? Them's the 3-d waves, e.g. representing the modes of vibration of our Mr. Sun. Below is a vibrating membrane, but you get the idea.

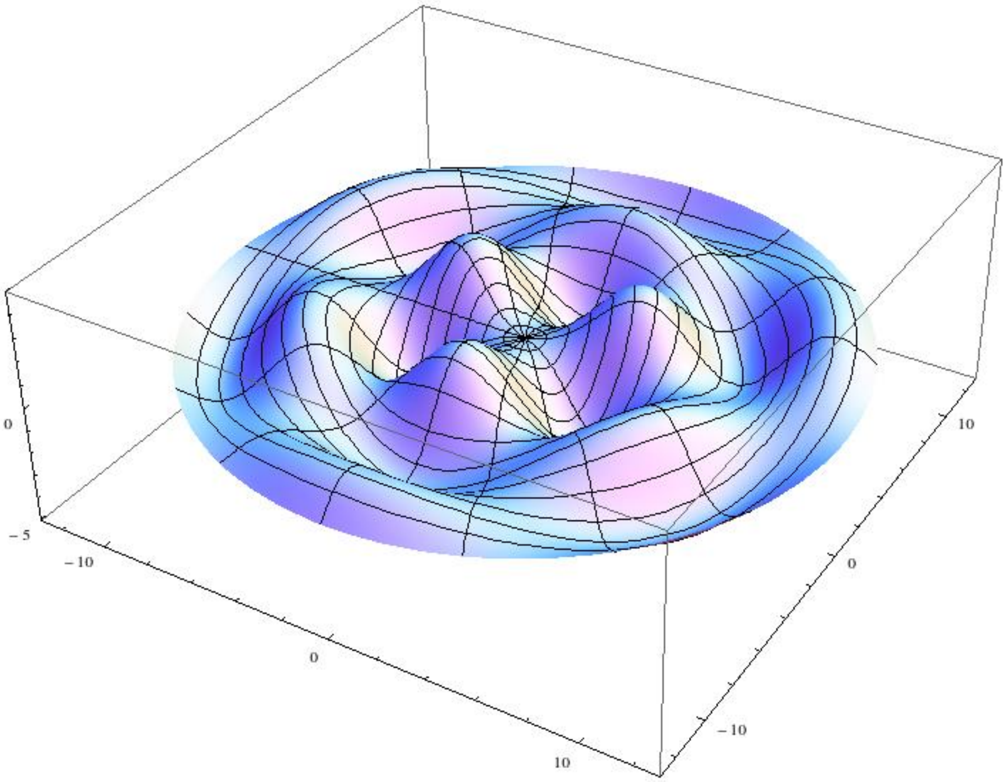


Figure: Vibrating drum Bessel function. Credit Slawomir Bialy. Bessel Function. Wikipedia.

Turns out wavefunctions in a uniform spherical potential well are a mess, no pattern. But the spectrum of a 3d quantum harmonic oscillator is nice and tidy. Here's the algebra. Hamiltonian has the same form expressed in 3d. Then it's all numerology.

$$H = \frac{\hat{p}^2}{2m} - \frac{1}{2} m\omega^2 \hat{r}^2$$

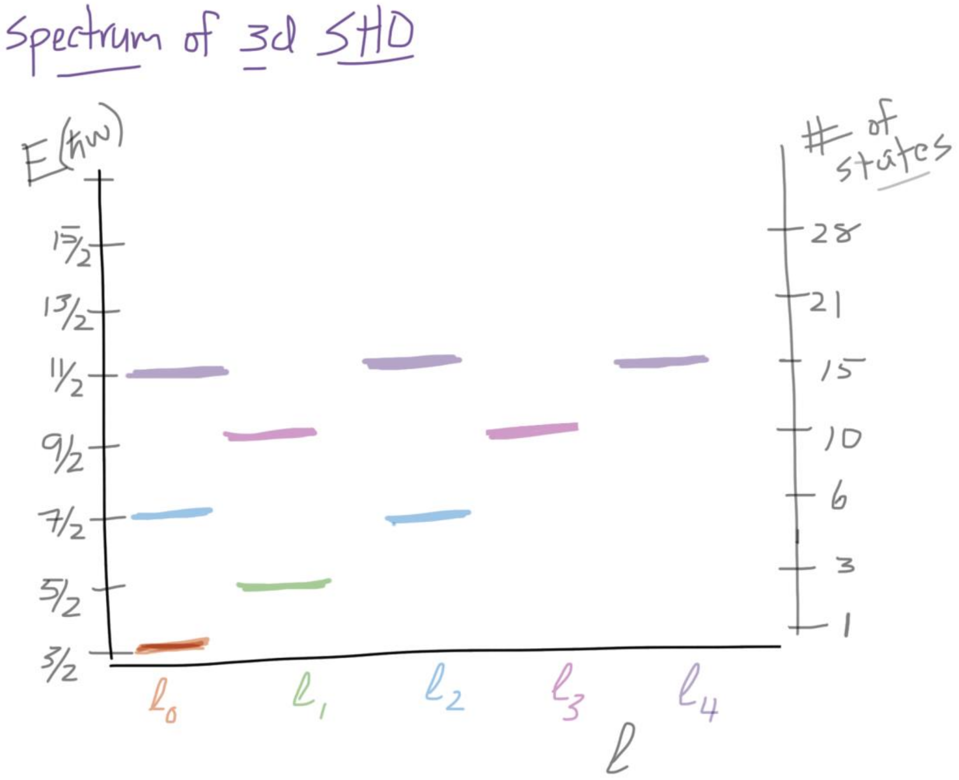
We build the spectrum of the 3d QHO like we did the 1d oscillator but with two more sets of ladder operators. And, note this, those operators form entangled states. The states in the QHO are tensor products. That's interesting; before we were entangling particles. Now we're entangling operators.

Start at the ground state, no angular momentum. $|\psi\rangle = |0\rangle$. In 3d that has energy $\frac{3}{2}\hbar\omega$. Now spin it up to one unit of angular momentum. There are three possible spin states, $a_x^\dagger|0\rangle$, $a_y^\dagger|0\rangle$, and $a_z^\dagger|0\rangle$ all with l_1 . Degenerate in energy $\frac{5}{2}\hbar\omega$.

l_2 gets trickier. Six possible states, combinations of the creation operators adding to two units of half-spin.

$$a_x^\dagger a_x^\dagger|0\rangle, a_x^\dagger a_y^\dagger|0\rangle, a_y^\dagger a_y^\dagger|0\rangle, a_y^\dagger a_z^\dagger|0\rangle, a_z^\dagger a_z^\dagger|0\rangle, a_x^\dagger a_z^\dagger|0\rangle$$

Well now. There's a problem. Six states totals energy $\frac{6}{2}\hbar\omega$. Ain't no such energy on the spectrum. We've got to split that degeneracy. Solution is five l_2 states and one l_0 all at $E = \frac{7}{2}\hbar\omega$. Onward and you get a spectrum that looks like



But hold on here. We built those six states in l_2 . How did we end up with a state in l_0 ? The answer (I think – Prof. Z. didn't address this directly) is entanglement. Entangle all those l_2 states and you get an isotropic system among them. Spherically symmetric. No preferred direction.

On then to the hydrogen spectrum. With all these tools available it's simple! Here I'd figured H would be a colossal maths challenge. Not so! It all spills out of the Hamiltonian of a central electric potential.

$$H = \frac{\hat{p}^2}{2m} - \frac{e^2}{\hat{r}} = \frac{\widehat{\hbar}^2}{2m\hat{r}^2} - \frac{e^2}{\hat{r}}$$

The Bohr radius immediately falls out. Set the potential and kinetic energies equal. Solve for radial distance.

$$a_0 = \frac{\hat{\hbar}^2}{2me^2}$$

Use that to solve the ground state energy: plug a_0 into the potential. Then ladder up the energy spectrum with the angular momentum operators! Same drill.

$$E_{nl} = -\frac{e^2}{2a_0}\frac{1}{n^2}$$

Surprising is how neat and tidy is the hydrogen spectrum. All kinds of degeneracy. You can see all the orbitals right there in the spectrum.

figure hydrogen spectrum

Lecture 24: Intro to perturbation theory

Here's interesting, and due to Feynman. The Feynman-Hellman theory. RPF figured it out as an undergrad.

Idea is that if you know the state of a system and you tickle it, say, with an extra potential you can add the perturbation to the operators on the initial state and get a close approximation to the perturbed state. Zwiebach's example here is fine splitting in the hydrogen spectrum due to the magnetic moment of the electron.

Feynman-Hellman says

$$\frac{\partial E(\lambda)}{\partial \lambda} = \left\langle \psi(\lambda) \middle| \frac{\partial \hat{H}(\lambda)}{\partial \lambda} \middle| \psi(\lambda) \right\rangle$$

Simple enough, and makes sense. You can check it out with the bra and ket algebra.

Now suppose $\hat{H}_{new} = \hat{H}_0 + \frac{\partial \hat{H}(\lambda)}{\partial \lambda} d\lambda = \hat{H}_0 + \lambda \hat{H}_0$. In the case of fine splitting, all we have to calculate is that second term. It's the change in energy due to interaction of spin orbital momentum with the magnetic moment of the electron. That's what we'll add as a perturbation.

$$V(L \cdot S) = -\frac{\hbar^2}{2mc^2}\hat{L} \cdot \hat{S}$$

where the orbital L and spin operators can be calculated from our previous calculations of the orbital radius and the fine structure constant. Not bad!

Lecture 24: Spin-orbit coupling

Question is: what happens when the physics includes multiple components. For example, the hydrogen atom has a central potential, the electromagnetic field anchored on the proton, plus the magnetic moment of the electron in that field. The motion of the electron induces an effective magnetic field, and μ_e interacts with that field. That's the situation introduced above. Now take a look at the states and their energies. It's pretty straightforward, really.

Represent the possible $|l, m\rangle$ orbital angular momentum states. l is total angular momentum; m is the z component. There are three possible states for $l = 1$.

$$|1,1\rangle, |1,0\rangle, \text{and } |1,-1\rangle$$

And there are two possible electron spin states.

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle \text{ and } \left|\frac{1}{2},-\frac{1}{2}\right\rangle$$

Altogether, then, there are six possible spin-orbital terms in the Hamiltonian.

$$|1,1\rangle \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle$$

$$|1,1\rangle \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle$$

$$|1,0\rangle \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle$$

$$|1,0\rangle \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle$$

$$|1,-1\rangle \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle$$

$$|1,-1\rangle \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle$$

The top and bottom states have energy $\frac{3}{2}\hbar\omega$. The middle four form a degenerate multiplet with energy $\frac{1}{2}\hbar\omega$.

It's all wavefunctions, but we can see the picture in a cartoon.

figure spin-orbit states

Lecture 26: The hydrogen spectrum

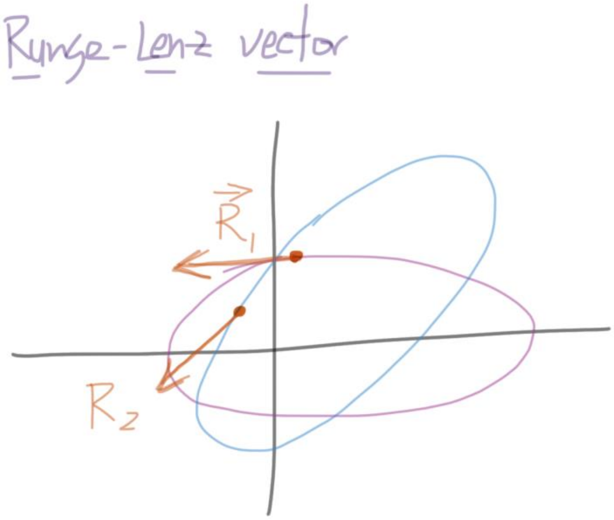
Done! Last of the lectures! And what a great lecture series! Prof. Zwiebach and his assistants do a marvelous job presenting the quantum mechanics.

Can I do the calculations? No, not well. I need to really sit down and practice. Do the problem sets. Do the exams.

But I think I have a whole lot better understanding of the concepts. I learned tons about vector spaces and operators. I learned tons about wavefunctions. I learned tons about Hamiltonians. I learned tons about complete sets of commuting operators. Tons.

Now the hydrogen spectrum seems a bunch of vector algebra. Blackboard after blackboard of equations. What it all comes down to, I think is:

1. Look at those 3d Bessel functions. In them you can see the angular momentum, total and z-component.
2. Consider the additional angular momentum included in the Runge-Lenz vector.



The what? Here's my take what's going on. Runge-Lenz $\vec{R}$ points along the major axis of an ellipse. It has constant magnitude, depending on $\hat{p}$. (I think . . . better check this.) If the orbital is precessing (why not?), then the precession, captured by Runge-Lenz, contributes to the total angular momentum. At least I think that's what's going on.

So what does it have to do with angular momentum? Precession. (My guess.) Suppose the electron has an orbital angular momentum $\hat{L}$ and spin $\hat{S}$ with total $\hat{J} = \hat{L} + \hat{S}$. Now suppose the whole system is precessing, major axis of the atom revolving around the nucleus. We have to include those effects, the precession, in calculations of angular momentum. Precession has to be included in the Hamiltonian. That takes the form $\hat{R} \times \hat{J}$. There's a whole bunch of commutation relations in there, and it all washes out as the spectrum. Review the lecture notes.

Meantime I'm off to do taxes and get ready for a Grand Canyon trip. Hasta luego!