General Relativity: the Geometry of Spacetime

In the last chapter, we introduced Einstein's general theory of relativity based on the principal of equivalence. In the present chapter, we probe further into Einstein's theory as a geometric model of gravity. Einstein described gravity as spacetime curvature: the presence of a mass curves spacetime, and other masses follow paths, geodesics, determined by the local curvature.

Quite the jargon – local curvature and spacetime and geodesics! But we proceed undaunted, first to understand the notion that "all physics is local," then what is spacetime? Next we consider how to measure the curvature of spacetime and especially Einstein's measuring tool, the metric. Finally, we look at "extremal time" and the calculations plotting how objects move.

Just as the special theory provides tools (e.g. the spacetime interval) to reconcile measurements made by inertial observers traveling at different velocities, the general theory provides tools (including the metric) to reconcile measurements made by observers who are accelerating relative to one another. Hence the general theory applies to all the nooks and crannies of the Universe, where planets orbit suns and plasma plummets into black holes. The general theory of relativity extends the special theory to include accelerated frames of reference.

Here, in brief, is where we've been and where we're headed:

Newton described gravity as a force, as if invisible cables pulled masses toward each other and held moons in orbit around planets, planets in orbit around suns. Such a description implied "action at a distance:" i.e. the presence of a mass here could affect the trajectory of another mass there. Newton himself rejected such an idea. "Hypotheses non fingo: I make no hypotheses about the nature of the force," he said. He only described the force.

$$F_{grav} = \frac{GMm}{r^2}$$

Whirl a ball on a string. Newton's gravity is analogous to the string holding the ball to your hand.

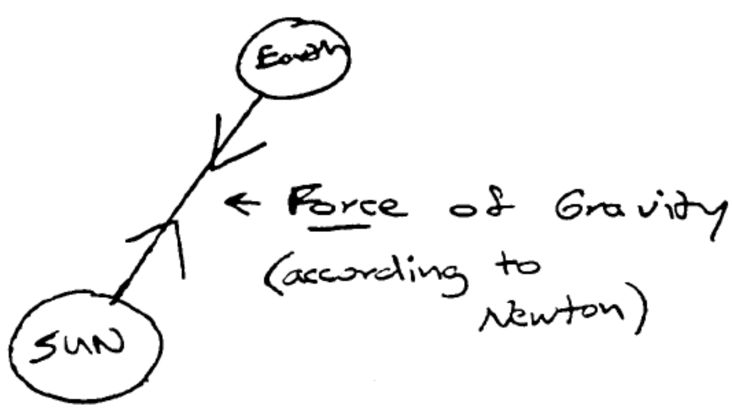

Figure 5.1. Newton's laws treated the force of gravity as action at a distance.

Field theory revised the classical model of gravity. A gravitational field permeates space around a mass, and interacting fields pull one mass toward another.

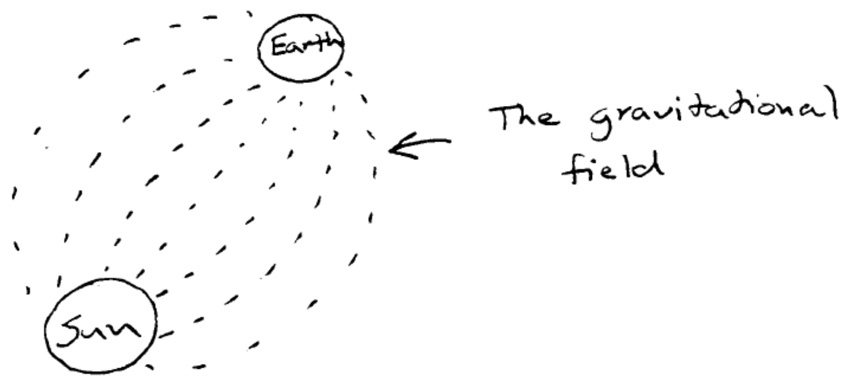

Figure 5.2. Gravitational fields generated by the two masses connect earth to sun.

Einstein extended field theory, arguing that a mass warps surrounding spacetime, and spacetime curvature, in turn, affects the motion of other nearby masses. Our efforts, indeed the whole of the general theory, are neatly summarized by John Wheeler: "Space acts on matter, telling it how to move. In turn, matter reacts on space, telling it how to curve."

Theoretical underpinnings

We have discussed the assumptions underlying the theory of special relativity: constant speed of light and uniform laws of physics for all inertial observers. Extending his theory to include accelerated frames of reference, Einstein insisted that all physics is local; i.e. an event must have a proximate cause. There is no "action at a distance;" an event here cannot cause an instantaneous event there. Such instantaneous cause and effect would violate the "universal speed limit" that no information can travel faster than light. Einstein insisted that masses receive their traveling orders directly and immediately from the spacetime in which they are embedded.

What is this "spacetime?"

For starters, time is the fourth dimension in our measuring system – the fourth component, together with the three spatial dimensions, we use to measure the Universe.

"Fourth dimension?" ¿Que pasa? We can visualize three dimensions easily. That's the world we live in: objects have height and width and depth – the chairs we sit in, the tree outside, the books we read. But a fourth dimension? Height, width, depth, and . . . ?

In reality, we experience four dimensions all the time. Three dimensional objects move: water flows down the river, Earth orbits the sun, our Galaxy moves in relation to other galaxies. Everything moves, and the measure of motion is change in position with time. Time is the fourth variable required to measure the Universe. In order to define the position of a ship at sea, we must indicate its latitude and longitude at a particular <u>time</u>. In order to land on the moon, we must arrive at a three-dimensional geometric location at the same <u>time</u> the moon is there.

Spacetime creeps into everyday conversation: we often talk of distance and time interchangeably: "it's fifty miles to town," or "it's an hour's drive to town." Astronomers regularly convert time to distance, distance to time using the speed of light, *c*, as the conversion factor. We say it's 93 million <u>miles</u> from Earth to sun, or about eight and a half light <u>minutes</u>. To the planetary system Alpha Centauri, it's about four light years, the distance light travels in four year's time.

In Einstein's theory, spacetime <u>is</u> the Universe. Space bends and warps according to the distribution of mass. Time changes, too, depending on the local mass distribution. Masses, in turn, follow paths determined by the local curvature of space and time, spacetime.

<u>A model of curved spacetime</u>

We can build a two dimensional model of spacetime. Stretch an elastic material across a basin 18 inches or more in diameter. Place marbles or bearings on the surface. Choose bearings with enough weight to depress the surface in the immediate vicinity, but not the entire surface. The marbles and bearings represent different masses in spacetime. Note how the masses "curve" spacetime. Now roll other marbles near them. Lightweight masses follow a curved path along the surface as they pass a heavier one. At the proper velocity and angle of approach, a light marble may even go into "orbit" briefly around a heavy bearing, until friction robs the marble of momentum and it falls into the potential well of the bearing.

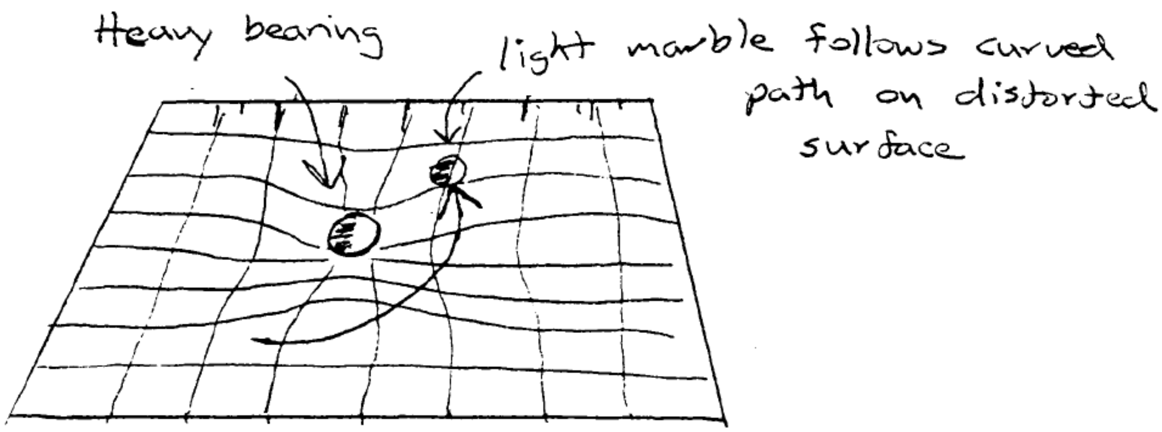

Figure 5.3. A two dimensional model of spacetime. Masses on a stretchy material distort the surface, and other masses passing nearby followed curved paths.

Heavier masses create deeper pits and appear to “pull” on other masses with greater force. But note that there is no force between the masses. They do not pull one another. Instead, the mass affects the local geometry of the surface, and the geometry, in turn, affects how other masses move.

(NB: With this model we use gravity itself to help model spacetime curvature (Einstein's gravity) , a circular but nevertheless useful demonstration. In the model, "real" gravity depresses the marbles and bearings into the surface to demonstrate the "geometric" gravity of general relativity.)

Curvature and common experience

In a way, it's obvious that spacetime is curved. Follow the trajectory of a baseball; it follows a curved path from bat to fielder's glove. Follow the path of an orbiting satellite; it traces an elliptical path around the earth. Trace the orbit of the earth around the sun, an ellipse again. From such it examples, it is clear that gravity somehow alters a mass's "normal," straight-line, inertial path. More precisely, the presence of one mass bends the expected straight-line path of another mass.

Einstein argued that moons (and orbiting satellites and home runs) are following straight-line paths in curved spacetime. An astronaut in a shuttle feels no force. Rather, the astronaut, in continuous free-fall around earth, feels no different than if he were drifting on a straight-line trajectory between the stars. In this regard, the orbiting craft **is** following a "straight-line" inertial path: unless the planet below him is spinning and "dragging" spacetime around with it (producing effects which the astronaut can measure – more on this effect below), there is no measurement, no experiment an astronaut can perform to determine whether he is in inertial flight between the stars or in orbit around the home planet.

We can measure the curvature of spacetime directly (at least in a thought experiment!): take four spheres of known rest mass, give them all the same initial velocity (most simply, zero velocity), and then measure how they move in relation to each other over time. The spheres trace the curvature of spacetime.

For example, imagine we place our four test masses in the vicinity of a star. They will fall toward the star, and as they fall they move in relation to each other.

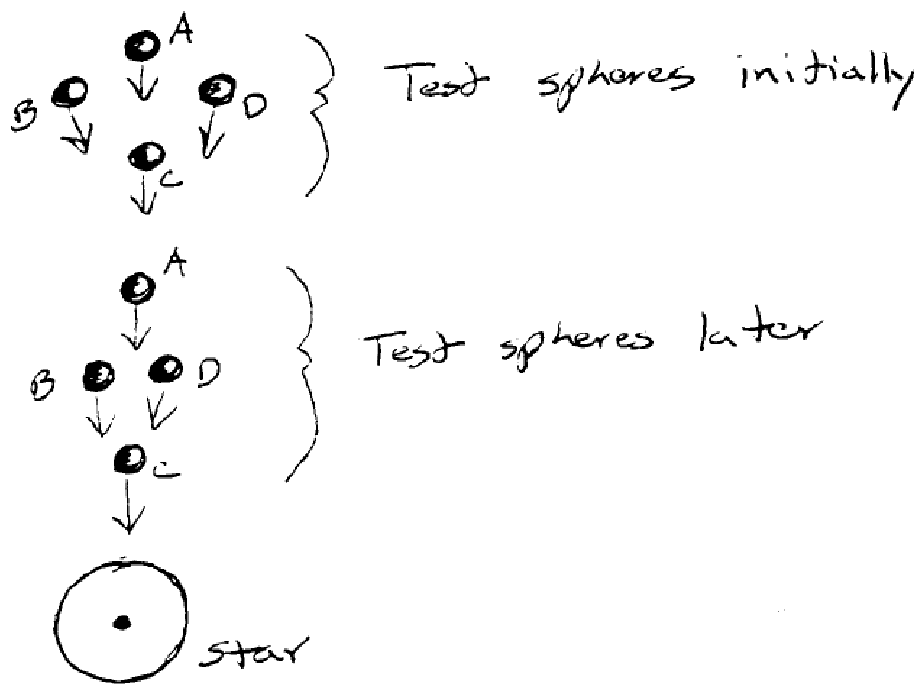

Figure 5.4. Test masses falling toward a large mass move relative to each other. Masses A and C, along the radius of fall, are stretched apart, while B and D, oriented perpendicular to the fall line, are "squeezed" together.

Spheres A and C move away from each other, because C is nearer the star, in a region of greater curvature (and in Newtonian terms feels a stronger "pull" of gravity, a "tidal force"). B and D move toward each other, since they fall along different radial paths toward the center of mass of the star (the center of curvature).

By measuring how the test masses move in relation to each other, we measure the local curvature of spacetime, even without reference to the star itself.

Spacetime reprise

It seems that spacetime itself has a structure. Test masses trace the outlines of that structure. There is something out there in "empty space."

In a way, that is not so surprising. The distinction between "mass" and "empty space" is not so definite as the terms imply. There is no distinct boundary, for instance, between Earth and outer space. Earth's atmosphere thins, there are fewer and fewer air molecules, at higher altitudes, but the atmosphere never "ends." Atoms, molecules, and subatomic particles drift through "empty space." Certainly, empty space buzzes with photons and neutrinos and perhaps other more exotic particles.

Even if we were to empty space completely – create a perfect vacuum, devoid of mass and energy (which is not possible, practically) – it would not be empty. The vacuum itself seethes with virtual particles, themselves carrying mass and energy. We shall discuss these virtual particles in more detail in a later chapter.

So the distinction blurs between mass and energy and "empty space." We can consider it all as a continuum, but with extra accumulations here and there. The stars and planets are like lumps in a pudding – same stuff as spacetime, just more of it

Notice that the shrinkage depends on the orientation of the meter stick relative to the acceleration, gravitational or otherwise. (The centripedal acceleration of our space station is radial, toward the hub.) Meter sticks oriented perpendicular to the acceleration shrink, while meter sticks oriented along the direction of acceleration do not. We should not be surprised at this dependence on orientation: according to the special theory of relativity, meter sticks shrink along the direction of motion, but meter sticks oriented perpendicular to the motion do not shrink.

<u>Measuring curvature: relative acceleration of test masses</u>

We can extend our intuitive argument about curvature. It seems clear that the stronger the gravitational field (i.e. the greater the mass producing the field, or the closer the test mass is to the mass generating the field), the greater the curvature. If the sun suddenly became ten times more massive, planet earth would have to increase its rate of revolution or else fall into the sun. The curvature of its orbit, measured in area swept out per unit time, increases. Or compare Mercury's orbit to Earth's: Mercury, closer to the sun hence in a region of greater spacetime curvature, orbits in a tighter ellipse and completes its circuit in less time.

Gravity curves orbits, and the greater the gravity the greater the curvature.

We can be more precise in our description of curvature. Imagine two test masses in orbit around a third, larger mas – two marbles, say, in orbit around the earth. Assume the test masses are so small that they exert negligible gravitational effects on each other and on the earth.

Separate the two masses north and south: place one in orbit around the equator, and launch the other at ten degrees latitude due north of the first. The masses, in their orbits, will follow "great circle" routes around the earth. The mass orbiting the equator will continue to follow the equator. The other mass will cross the equator twice on each orbit, oscillating between ten degrees north latitude and ten degrees south. From the perspective of the mass on the equator, the second test mass is oscillating back and forth in its orbit, alternately approaching, crossing paths, then receding. The test masses are always accelerating relative to each other.

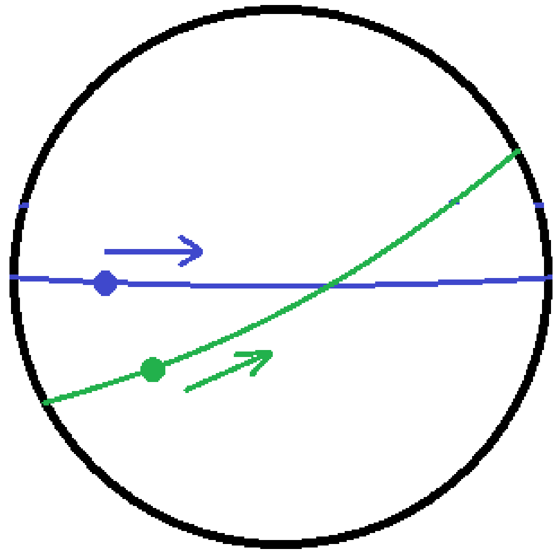

Figure 5.5. Satellites on different geodesics accelerate relative to each other.

We can use the relative acceleration as a measure of curvature. Placed in orbit, i.e. in free-fall trajectories, at some distance, *r*, from the center of the earth, the masses would accelerate at some rate toward each other. Placed in orbit around the moon at distance *r* from the center of the moon, the relative acceleration would be much less, since the moon has less mass and generates less curvature in surrounding spacetime. Placed at distance, *r*, from the center of a black hole, the relative acceleration would be much greater.

The test masses provide a means of measuring the effects of gravity independent of reference to the mass responsible for the gravity. The test masses are measuring the local curvature of the spacetime through which they travel. (See Appendix for other methods of measuring curvature.)

Stretchy meter sticks

Finally we consider Einstein's more formal method of measuring curvature, the metric. The metric is the spacetime interval refined to accommodate curvature and returns us to familiar territory, measurement with clocks and rulers.

Draw a circle on a sheet of paper. The circle has a circumference given by (and measurable as) $2\pi r$, where r is the radius of the circle, and the circle has an area of πr^2. Now drape the paper disk over a globe, center of the circle on the north pole and circumference toward the equator.

When you drape the circle over the globe, you must trim wedges out of the disk in order to fit it to the surface. The radius of your circle stays the same, but the globe's curvature changes the geometric relation between radius and circumference and between radius and area. The circumference of the circle draped on the globe is now <u>less than</u> $2\pi r$, and the area of the circle is <u>less than</u> πr^2. Alternatively, we can describe these geometric effects of curvature in terms of "radius excess:" the radius of our circle, measured on the surface of the globe, is <u>longer</u> than we would expect for the measured circumference and area.

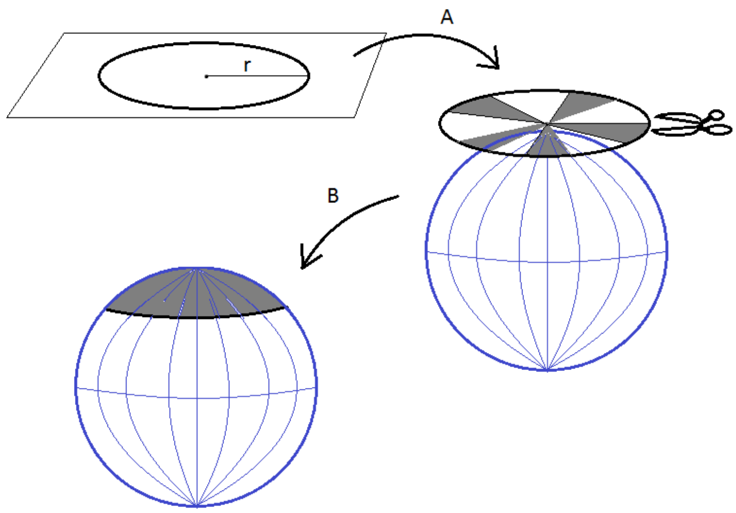

Figure 5.6. In order to drape a circular disk on a sphere, you have to cut wedges from the disk.

Curvature changes geometry. On the surface of a globe, we can no longer describe circles with the familiar Euclidean equations.

It turns out we can describe this change in geometry as a change in our measuring tools: the radius excess in curved space is equivalent to a change in our meter sticks in flat space. Suspend judgment for a moment, and imagine that the length of meter sticks can change with location. We return to work on a flat table top, drawing circles and measuring circumferences. It's an unusual table top, however, cold in the center, hot toward the edges. So our metal meter stick changes length as we move it around the table, measuring circles. The meter stick shrinks slightly in the center of the circle, where the metal contracts in the cold. It lengthens slightly toward the circumference of the circle, as heat expands the metal. So when we measure the radius and circumference of our circles, drawn on flat paper on a flat table, we find a radius excess! Because it doesn't take as many meter sticks to go around the circumference, where the meter sticks are longer, the measured circumference is smaller than we would expect for the measured radius. It's as if our flat circle had been draped over a sphere.

So, Einstein realized, we can describe curvature as a change in measuring tools, a change in the metric.

The metric

Clever argument, stretching meter sticks to mimic curvature. But what stretches clocks and meter sticks out there in real spacetime? We adduce the principle of equivalence and demonstrate the change in measuring tools with a thought experiment based on the argument that rotational acceleration mimics gravity.

It's the year 2021, and mankind has constructed a complex space station, a giant spinning wheel with several spokes radiating out from the hub to three concentric circles at different radii from the hub. The centrifugal force at the middle circle, i.e. the acceleration of the rim pressing up against an inertial body, is exactly 9.8 m/sec/sec, Earth's gravitational acceleration. Here, in this circle, are the living quarters. The inner circle, experiencing less rotational acceleration, therefore less effective "gravity," is the storage area. The outer circle, experiencing greater rotational acceleration and greater equivalent "gravity," provides a training facility for astronauts scheduled to visit the giant planets.

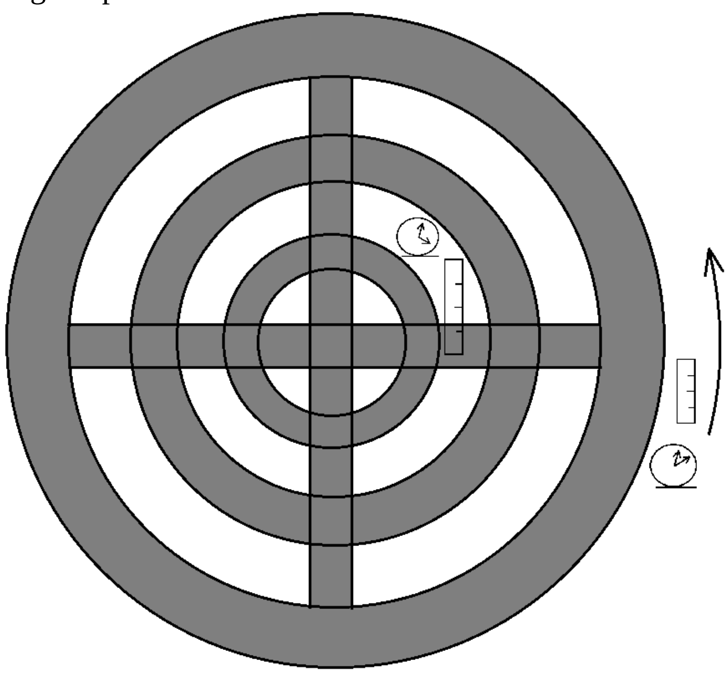

Figure 5.7. Meter sticks on outer rim of spinning space station are shorter and clocks tick slower than meter sticks and clocks on the inner circle.

We perform experiments in our station to determine the effects of acceleration and therefore, by the principle of equivalence, the effects of gravity, on clocks and meter sticks. All clocks have been synchronized and meter sticks cut to uniform length at the hub and transported to the various circles. We shall compare the meter sticks and clocks at the various circles with a standard clock and meter stick in the hub.

First, place meter sticks end to end around the circumferences of each of the three circles, measuring the various circumferences. Lo and behold! The measured circumferences are greater than expected, and the deviation from the expected circumference increases as we progress outward from the hub. (The expected circumference is calculated by the Euclidean formula $= 2\pi r$.)

What's happened?

First a mathematical note. Rotating objects experience a centripedal acceleration and a centrifugal force. The acceleration is calculated by

$$a = \frac{v^2}{r}$$

where v is the tangential velocity of the spinning object at the distance r from the center. The v^2 term dominates, so as we move farther from the center the centripedal acceleration increases.

Placed along the circumference of our rotating space station, a meter stick experiences a relativistic length contraction. The farther the meter stick from the hub, the greater its relative velocity and centripedal acceleration, and the more it shrinks. Meter sticks shrink in an accelerated frame of reference, so by the principle of equivalence, gravitational acceleration must shrink meter sticks, too.

Now check the clocks. As seen by an observer at relative rest in the hub, the clocks out in the circles slow down because of relativistic time dilation, and the farther clocks, moving relatively faster in their rotation, tick slower than the inner clocks. Clocks in an accelerated frame of reference slow (the interval between ticks increases). By the principle of equivalence, gravitational acceleration must also slow clocks, and the deviation from the fiducial rate is proportional to the acceleration.

These thought experiments indicate that gravity slows clocks and shrinks meter sticks. Gravity changes space and time. By analogy with our previous arguments regarding radius excess on a globe, gravity curves spacetime. We can, therefore, describe gravity in terms of geometry: the greater the curvature, the greater the gravity.

What generates curvature?

What generates curvature? What bends the paths of planets and baseballs?

Obviously, gravitational curvature originates in mass, but more than mass contributes. As we would expect from our consideration of special relativity, mass is more than mass!

Since

$$m = \frac{E}{c^2} \quad ,$$

energy must also contribute to gravitational curvature. And there are a number of sources of energy:

- Consider a star, for instance. If we weighed all the nuclei and electrons in the star, we would derive a certain mass. But those nuclei and electrons move at high velocities; they have kinetic energy, measured by the temperature of the star. And that kinetic energy must be included in the gravitational mass of the star.

- Usually stars are spinning. So we must include the energy of rotation.

- More than that, the star emits radiant energy, and (as we shall see in a subsequent chapter) much of that radiation is trapped in the core of the star. That radiant energy must also be included in the gravitational mass of the star.

- Then there is the self-gravity of the star itself. The overlying layers of the star press down on the core, and that gravitational energy must be included also in the total mass of the star.

- Finally, we must consider who is measuring the mass. An observer at rest relative to the star will measure one mass (including all the considerations above). An observer moving relative to the star will measure a greater mass, depending on her velocity, because of relativistic effects.

Following Wheeler, we can bundle all of these variables in the vector quantity momenergy – the relativistic momentum + energy of a system. Einstein included all of these variables in the stress-energy tensor. His model of curvature is elegantly summarized in the gravitational field equation: $G = 8\pi T$, where G, the Einstein curvature tensor, measures gravitational curvature produced by T, the stress-energy tensor of a system, its momenergy. Mathematically, T is a four-by-four matrix including contributions to all four spacetime components (t, x, y, z) of momenergy from each of the other components.

The metric

If clocks and meter sticks change in a gravitational field we must accommodate those changes in the spacetime interval: the measure of the spacetime separation between two events will depend on the spacetime curvature in which the events are embedded. In a gravitational field, as on the surface of a sphere, the shortest distance between events is no longer a straight line but a geodesic.

To define the spacetime interval in curved spacetime, let's first simplify by reducing our purview from four dimensions to a two-dimensional surface. Draw a Cartesian (x, y) coordinate grid on a sheet of rubber. The initial "flat" (all x coordinate lines parallel, all y coordinate lines parallel and perpendicular to x) surface represents empty space – "Euclidean" space – devoid of any momenergy. Now drop a steel bearing on the sheet: the coordinate grid is distorted in proportion to the mass of the bearing.

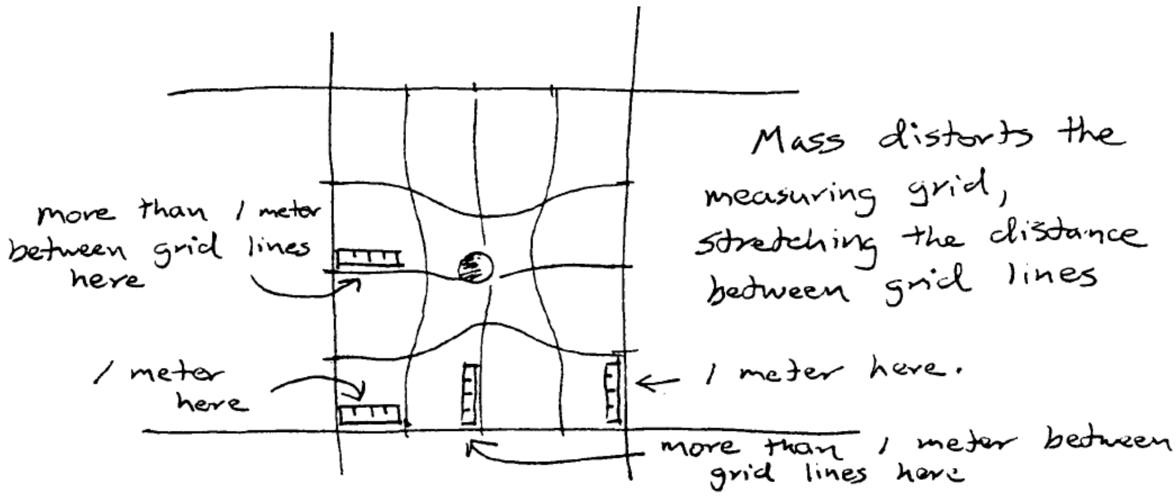

Figure 5.8. Mass, e.g. a steel bearing, on a stretchy surface creates a pit in the surface which distorts grid lines drawn orthogonal and equidistant on the original flat surface.

We can measure the curvature. We know the distance A to B on an unstretched sheet. If we measure the distance A to B on the distorted sheet with an unstretched ruler, we can determine how the *x* component of distance varies according to our position on the *x* and *y* axes and also how the *y* component of distance varies according to position on the *x* and *y* axes. Mathematically, we can determine

dL_x/dx

dL_x/dy

dL_y/dx

dL_y/dy

where the L_α represent grid length along the *x* or *y* axes. Note that this device, viewed relative to the grid, measures the change in length of the meter stick (or clock) as we move from event to event.

We proceed now from 2D to 4D spacetime. In flat spacetime (empty space, no nearby masses, no gravitational field) the spacetime interval describes the separation between two events:

$$d\tau^2 = dt^2 - \left(dx^2 + dy^2 + dz^2\right)$$

In curved space, distance measurements (the *dx*, *dy*, and *dz*) themselves change, as on our stretched 2D sheet, and clocks (*dt*) also change. In order to describe the spacetime separation between two events, then, we must include terms that describe how much our measuring devices stretch or shrink between events.

$$d\tau^2 = g_{\alpha t} dt^2 - \left(g_{\alpha x} dx^2 + g_{\alpha y} dy^2 + g_{\alpha z} dz^2 \right)$$

The *g*'s, the metric coefficients, describe how much each meter stick or clock changes if it is displaced in a certain direction. The α's are shorthand for each possible direction – *x*, *y*, *z*, and *t*. So g_{yx}, for example, describes how much the meter stick oriented along the *x* direction will change if it is displaced along the *y* direction, and g_{zt} describes how much a clock changes if it is displaced along the *z* direction. Since events are separated in space and time, we require all the metric coefficients for a complete description of the separation between those events. Since spacetime is just the sum of all events in the universe, these metric coefficients provide all the information we need to map it.

In flat spacetime, in the absence of any mass-energy, the *g*'s all equal 1, and the equation reduces to the familiar spacetime interval. In the vicinity of a spherical, symmetric, non-rotating mass, the time coefficient, $g_{\alpha t}$, changes according to the central mass and distance from the mass: if a clock is located along the *z* axis relative to the mass, $g_{zt} = (1 + 2M/r)$, where *M* is the mass/energy and *r* is the distance from the clock to the center of mass. For the *x* meter stick located at distance *r* along the *z* axis relative to a spherically symmetric, nonrotating mass, $g_{zx} = (1 - 2M/r)$.

Consider another perspective using polar coordinates. Polar coordinates simplify the description of spacetime around a spherically symmetric object, such as a star or black hole. The metric, in polar coordinates, is

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} - r^2 d\Omega^2$$

where τ is the time on a free-fall clock, *t* is the time measured by a distant observer, *M* is the mass of the object toward which the mass is falling, *r* is the clock's radial distance from the mass, and $d\Omega$ is the radial change in angle if the clock has a component of motion tangential to the mass. This is the Schwarszchild metric, formulated by Karl Schwarszchild, an early pioneer in understanding general relativity.

We will usually be interested only in the radial terms – how the metric changes with radial distance from the central mass – so we will ignore the last term, which concerns tangential displacements at distance r.

$$d\tau^2 = \left(1 - \frac{2GM}{r}\right) dt^{\,2} - \left(\frac{1}{1 - {}^{2GM}/_{r}}\right) dr^2$$

This metric allows us to describe the curvature – changes in clocks and meter sticks – in the region around the mass as a function solely of r, the distance from the mass, without regard to x, y, z direction. All directions away from the center of mass are equivalent.

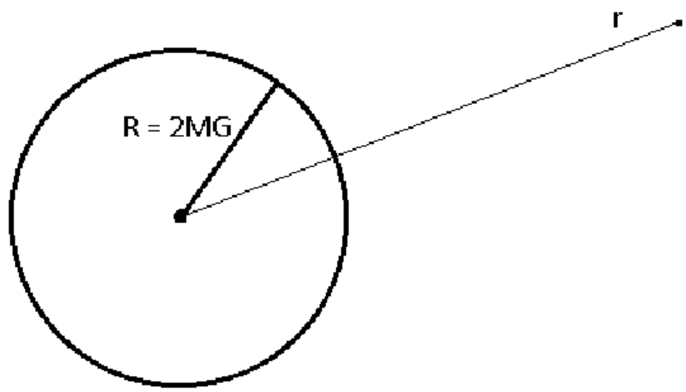

Figure 5.9. Coordinates for the Schwarszchild metric. R is the radius of the spherically symmetric mass, e.g. black hole, and r is the distance from center of mass to an object outside the black hole.

To analyze spacetime near a rotating mass, we must take into account other effects as described by the stress-energy tensor. For example, a rotating mass "drags" nearby spacetime around with it. We will consider such issues further in a subsequent chapter.

Fine. The equation looks authentically metric, a spacetime measure of separation. But whence those coefficients with $\left(1-\frac{2M}{r}\right)$?

Consider the familiar time dilation equation from special relativity.

$$t=\frac{\tau}{\sqrt{1-v^2/c^2}}$$

Reinterpret τ as the time interval on the free-fall clock and t as the interval measured by an outside observer.

$$\tau=t\sqrt{1-v^2/c^2}$$

Now, what velocity measures the mass of an object? The escape velocity! We can measure the mass by measuring the velocity with which we have to launch a rocket in order to escape the object's gravitational pull.

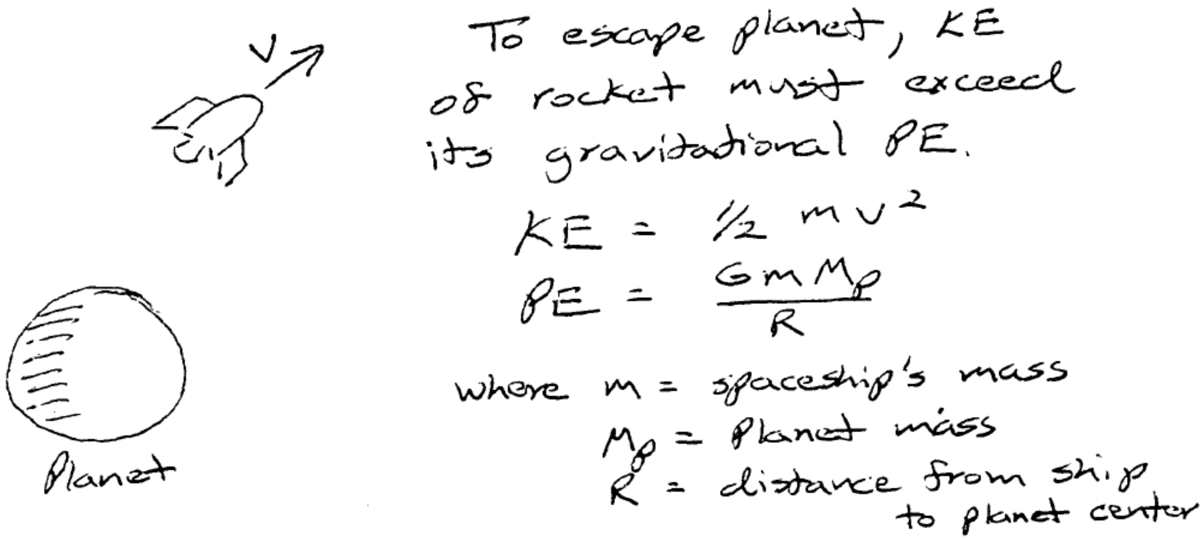

Figure 5.10. In order to escape completely from planet, the rocket must attain sufficient velocity, the escape velocity.

Solving for the minimum escape condition where KE just equals PE, we find the escape velocity

$$v_{esc} = \sqrt{\frac{2GM}{r}}$$

Substitute into the time dilation equation:

$$\tau = t\sqrt{1 - \frac{\left(\sqrt{2GM/r}\right)^2}{c^2}} = t\sqrt{1 - 2GM/rc^2}$$

By convention in relativistic calculations, we let G and c equal one, so

$$\tau = t\sqrt{1 - 2M/r}$$

In the condition of small intervals $d\tau$ and dt, and squaring both sides, we have the expression for the time component of the metric. A similar argument gives the spatial component with dr.

Geometrodynamics: describing how objects move through spacetime

The metric measures spacetime separation between events, and all observers agree on the spacetime interval between two events. Now we need a device that allows us to predict how an object will move through spacetime. One of the triumphs of Newtonian physics, after all, was the capacity to predict: given initial positions and velocities, Newton could predict future positions and velocities of planets and moons based on his law of gravitation. Masses in the solar system are generally below the limits at which relativistic effects become noticeable, so Newton's laws allow accurate descriptions. But at relativistic masses and velocities, as in the vicinity of black holes and neutron stars, Newton's laws fail.

It turns out that the "marching orders" for objects in spacetime are direct and simple: "follow the path of extremal time." That is, masses moving through spacetime follow free-fall trajectories, geodesics, between events, along which paths a maximum of time passes on their on-board clocks. Following any other path between the same events and less time would pass.

This seems strange, at first glance. In our everyday experience, the shortest distance between events is the path of least time: a direct flight from San Francisco to New York takes less time than the route via Houston. But consider the following argument:

Draw a spacetime diagram, time on one axis, distance on the other. In the frame of reference of an object at rest, there is no displacement along the distance axis, but time always passes so the object moves along the time axis. Suppose, for example, you are at rest in your chair, reading, between the hours of seven and eight as measured by the wall clock. Time passes, but you're not traveling anywhere in space. Your watch and the wall clock both tick a certain number of times, say 3600 ticks, one tick each second, between event A, 7:00 at rest in your chair, and event B, 8:00 at rest in your chair.

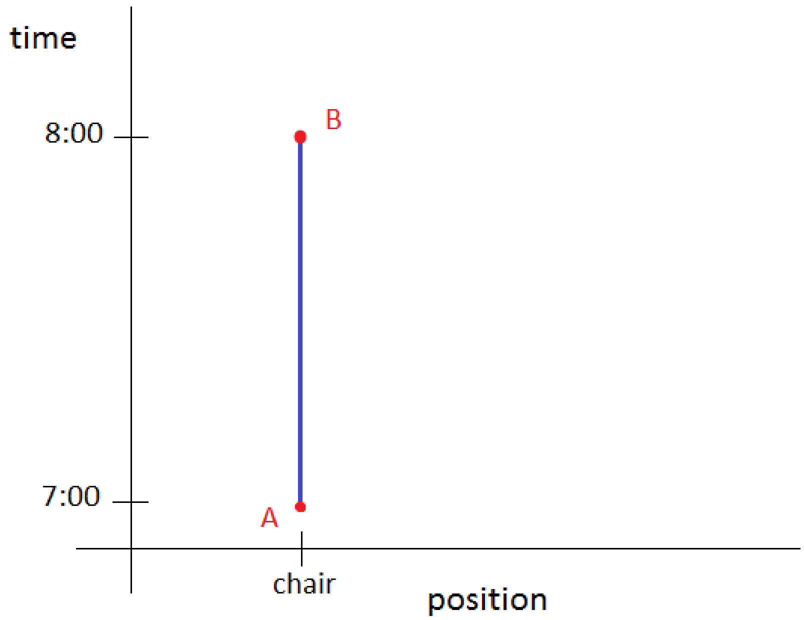

Figure 5.11. Spacetime diagram of you at rest in your chair between 7:00 and 8:00 PM.

Suppose, on the other hand, at 7:01 your alter-ego dropped your book and took the dog for a brisk walk around the park, returning to your chair at 7:59 wall-clock time. You and your alter ego compare watches at event B, 8:00 wall-clock time. You find that your alter-ego's dog-walking watch, affected by relativistic time dilation, only recorded fewer ticks between the same two events.

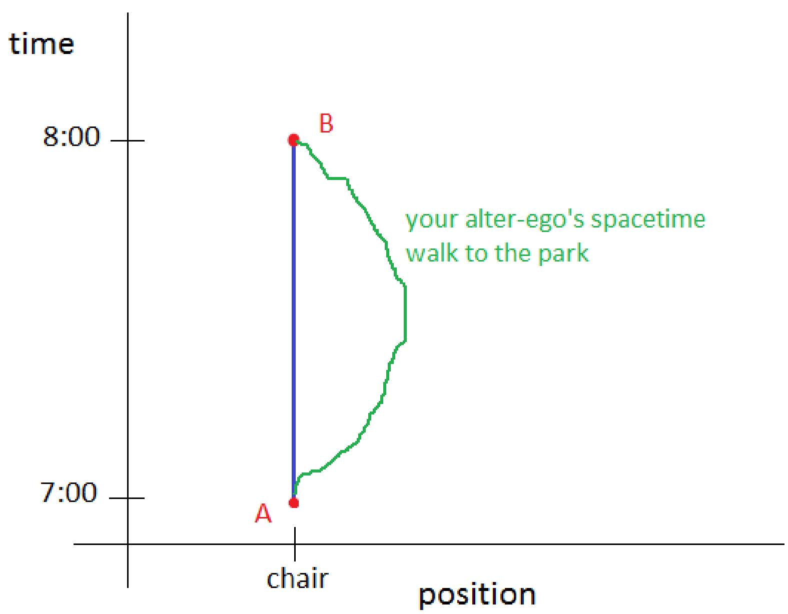

Figure 5.12. Spacetime diagram showing your geodesic path (vertical line along the time axis) vs. your alter-ego's walk to the park.

You, at rest in the chair, have followed the spacetime geodesic between events (ignoring the acceleration of the earth's surface pressing against your feet). Your alter-ego, on the other hand, has traveled farther between events, around the park with the dog, departing from the geodesic. That departure is evidenced by the deficit in ticks on alter-ego's watch.

Clocks on geodesics tick off maximal time. Clocks that accelerate off the geodesics slow down. So the traveling orders in spacetime, the orders by which masses follow their free-fall trajectories, demand paths of maximal time.

This "principle of maximal time" provides a link back to familiar Newtonian physics, a check on our reasoning. Following Feynman, we can show that, in the low-energy, low-velocity approximation, the principle of maximal time gives us Hamilton's "principle of least action:"

Follow an object, say a baseball in flight, in free-fall along its trajectory. (We'll ignore the effects of air resistance.) As the baseball rises in its arc, its clock – its "proper time" – will tick slightly <u>faster</u> due to the change in gravitational potential. At any event (time and position) on its flight, the proportional change in proper time due to gravitational potential $= gh/c^2$, where h is the height above the ball park. On the other hand, the ball's clock <u>slows</u> because of its velocity: at each point along the trajectory, the proportional change in proper time due to velocity $= v^2/2c^2$, in the low velocity approximation . If we sum the proper time <u>excess</u> due to height in the gravitational and the proper time <u>deficit</u> due to velocity all along the ball's path, the integral must be a maximum (by the principle of maximal time).

$$\int \left(\frac{gh}{c^2} - \frac{v^2}{2c} \right) dt$$

is a maximum.

Now multiply this integral by $-mc^2$.

$$\int \left(-mgh + \frac{mv^2}{2} \right) dt$$

This is Hamilton's "principle of least action," which states that the time integral of kinetic energy minus potential energy of an object in free fall is a minimum, and Hamilton's principle reduces directly to the familiar Newtonian equations of motion, describing motion in terms of mass and acceleration. As we would hope, the description provided by general relativity includes and is compatible with Newton's laws at low velocities and low energies.

<u>How general relativity differs from Newtonian physics</u>

As with special relativity, we don't notice the effects of general relativity directly in our daily lives. We can't look west and see the back of our head; Earth's mass isn't great enough to curve the path of light noticeably. And we don't have to plan to arrive early if we make a reservation at a restaurant on the top floor; the clocks in the restaurant aren't running more than a few billionths of a second faster than our clocks at ground level.

However, on astrophysical scales – around stars like our own sun and behemoths like neutron stars and black holes – relativistic effects predominate, and the consequences differ radically from the predictions of Newtonian physics.

For example, while Newton allows for increasingly strong gravitational fields around stars of greater and greater mass, general relativity predicts runaway collapse when stars reach a critical mass. According to general relativity, gravitational curvature increases the pressure and the energy density inside a star, which increases the curvature, which increases the density . . . There's a self-augmenting feedback, and the star collapses on itself. (More on such stars in a subsequent chapter.)

As another example of the difference between the predictions of Newtonian physics and general relativity, consider the effects of angular momentum. In Newtonian physics, a star is effectively separated from the surrounding space. In general relativity, however, the curvature of spacetime is generated by and connected to the mass/energy.

As a final example of departure from Newton, general relativity predicts gravity waves, waves of changing curvature, generated by accelerating masses, propagating through spacetime. If the local curvature changes suddenly, as when binary neutron stars finally collide, the change in curvature propagates outward through spacetime, like waves across the surface of a pond from the point where a pebble splashed into the water.

General relativity connects spacetime intimately to mass-energy. There is no boundary between "mass" and "empty space." To repeat Wheeler's observation: "Mass tells spacetime how to curve, and spacetime tells mass how to move," a continuous interplay, the curvature modifying the mass-energy modifying the curvature.

Summary

The general theory extends relativity theory to include accelerating frames of reference, encompassing all of nature. It is Einstein's theory of gravity.

Einstein interprets gravity as geometry: mass-energy curves spacetime, and the curvature determines how mass moves. Moreover, physics is local: local spacetime curvature tells mass how to move. There is no "action-at-a-distance," no distant mass there bending the trajectory of mass here.

Any of several methods that can be used to measure curvature:

- The local curvature of spacetime can be measured by the relative acceleration of one free-falling test mass toward another.

- Curvature can be measured by the metric coefficients: curved geometry is analogous to a change in meter sticks and clocks. The metric is Einstein's choice for describing curvature.

We can predict the trajectories of particles in a gravitational field according to the principle of extremal time: objects in free-fall follow trajectories (geodesics) which produce the greatest lapse of proper time.

In the theories of special and general relativity, Einstein unified several previously disparate concepts:

- Prior to Einstein, scientists believed mass, length, and time were invariant quantities, and unrelated. Einstein showed those quantities are, in fact, inter-dependent according to their state of motion. Space and time are related: events must be measured in terms of space and time together, and the spacetime interval allows any observer to measure any sequence of events and correlate his measurements with any other observer.
- Classical physicists assumed mass and energy were distinct quantities. Einstein showed they are inter-convertible.
- Finally, Einstein revised the concept of force. We can understand the force of gravity in terms of geometry, the curvature of spacetime.

In the following chapters, we shall trace the influence of these unifications on the study of subatomic particles and the forces of Nature.