We have found clues that the string spectrum includes particles familiar in the Standard Model. (Note, however, that no string theory yet reproduces the Standard Model exactly. More on that later.) And – what's most intriguing to theorists – string theory includes gravity at its very foundation. Next we explore extra dimensions. String theory requires extra spatial dimensions for consistency, and those extra dimensions allow multiple versions of the theory.

Extra dimensions

Strings live in extra dimensions. Open strings require 26 dimensions for their wiggles, closed strings 10, including the four familiar dimensions of spacetime. Where did this crazy notion come from?

Extra dimensions are hidden in the equations. To reveal them, our argument will require the expression of the exponential function $e^{-n\varepsilon}$ as a series. Here's a brief review of series.

We want to re-write a function as a polynomial. For convenience, we'll focus on the behavior of the function near zero. What value in the polynomial will give us the same value as the function at zero? Clearly, that is the value $f(0)$, the value of the function at zero. So the first term in the polynomial, p, will be

$$p(0) = f(0) \tag{11.1}$$

We also require that the first derivative of the polynomial, evaluated at zero, must equal the first derivative of the function. That will be the case if

$$p(0) = f(0) + f'(0)x \tag{11.2}$$

To check that this meets our requirement, take the derivative of both sides.

$$\frac{d}{dx}p(0) = \frac{d}{dx}\big(f(0) + xf'(0)\big) = 0 + f'(0) \tag{11.3}$$

Similarly, we require that the second derivative and higher order derivatives of the polynomial equal the second and higher order derivatives of the original function. This will be the case if our polynomial is of the form

$$p(0) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f'''(0)x^3 + \ \dots \frac{1}{n!}f^{(n)}(0)x^n \tag{11.4}$$

where n is the last term in the series. (You can check by taking the various derivatives. Note that constants disappear when you take a derivative, and terms with remaining factors of x disappear because they evaluate to zero at $x = 0$.)

For the particular function $e^{-\varepsilon}$, the polynomial is of the form

$$e^{-\varepsilon} = 1 - \varepsilon + \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{6} + \ \dots \tag{11.5}$$

where the derivatives are taken with respect to ε .

OK. So what is this $e^{-n\varepsilon}$? We are ready to study the energy spectrum of open strings as polynomial functions. The polynomials, we shall discover, require that the energy is distributed through 26 dimensions.

Start with the energy equation

$$E = \hbar\omega \tag{11.6}$$

We've established $\omega = n$, the mode number, and as usual we let $\hbar = 1$. Then

$$E = n = \sum_{n=1}^{\infty} n e^{-n\varepsilon} \tag{11.7}$$

where ε is an infinitesimally small number such that the equality applies.

Why on earth do this? So that we can rewrite the summation in such form that we can derive an actual value for the energy. Follow on.

$$\sum_{n=0}^{\infty} n e^{-n\varepsilon} = \frac{-\partial}{\partial\varepsilon} \sum_{n=1}^{\infty} e^{-n\varepsilon} = \frac{-\partial}{\partial\varepsilon}(e^{-\varepsilon} + e^{-2\varepsilon} + e^{-3\varepsilon} + \ \dots) \tag{11.8}$$

$$= \frac{-\partial}{\partial\varepsilon} e^{-\varepsilon}(1 + e^{-\varepsilon} + e^{-2\varepsilon} + e^{-3\varepsilon} + \ \dots) = \frac{-\partial}{\partial\varepsilon} e^{-\varepsilon}\left(\frac{1}{1 - e^{-\varepsilon}}\right)$$

A bit more review required here. Where did the $\left(\frac{1}{1-e^{-\varepsilon}}\right)$ come from? Multiply the series

$$S = 1 + e^{-\varepsilon} + e^{-2\varepsilon} + e^{-3\varepsilon} + \ \dots \ e^{-n\varepsilon} \tag{11.9}$$

by $e^{-\varepsilon}$.

$$e^{-\varepsilon}S = e^{-\varepsilon} + e^{-2\varepsilon} + e^{-3\varepsilon} + \ \dots \ e^{-n\varepsilon} + e^{-(n+1)\varepsilon} \tag{11.10}$$

Subtract (11.10) from (11.9).

$$S - e^{-\varepsilon}S = S(1 - e^{-\varepsilon}) = 1 - e^{-(n+1)\varepsilon} \tag{11.11}$$

The last term on the right is infinitesimally small, so we drop it. Divide by $(1 - e^{-\varepsilon})$ and we recover the last factor in equation (11.8).

$$S = \frac{1}{(1 - e^{-\varepsilon})} \tag{11.12}$$

Onward, from equation (11.8). We use our transformation rules (11.4) to convert the exponential function $e^{-\varepsilon}$ into a power series.

$$\frac{-\partial}{\partial\varepsilon} e^{-\varepsilon}\left(\frac{1}{1 - e^{-\varepsilon}}\right) = \frac{-\partial}{\partial\varepsilon}\left(\frac{1 - \varepsilon + \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{6} + \ \ldots}{1 - \left(1 - \varepsilon + \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{6} + \ \ldots\right)}\right) \tag{11.13}$$

$$= \frac{-\partial}{\partial\varepsilon}\left(\frac{1}{\varepsilon}\right)\left(\frac{1 - \varepsilon + \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{6} + \ \ldots}{\left(1 - \frac{\varepsilon}{2} + \frac{\varepsilon^2}{6} + \ \ldots\right)}\right)$$

To proceed, we reverse our function-to-polynomial transformation rule. Equation (11.13) is of the form

$$\frac{-\partial}{\partial\varepsilon}\left(\frac{1}{\varepsilon}\right)\left(1 - \varepsilon + \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{6} + \ \ldots\right)\left(\frac{1}{\left(1 - \frac{\varepsilon}{2} + \frac{\varepsilon^2}{6} + \ \ldots\right)}\right) \tag{11.14}$$

$$= \frac{-\partial}{\partial\varepsilon}\left(\frac{1}{\varepsilon}\right)\left(1 - \varepsilon + \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{6} + \ \ldots\right)\left(\frac{1}{1 - s}\right)$$

where

$$s = \frac{\varepsilon}{2} - \frac{\varepsilon^2}{6} + \frac{\varepsilon^3}{24} - \cdots \tag{11.15}$$

But

$$\left(\frac{1}{1 - s}\right) = 1 + s + s^2 + s^3 + \ \ldots \tag{11.16}$$

by the same logic that gave us (11.12) above. So

$$\frac{-\partial}{\partial\varepsilon}\left(\frac{1}{\varepsilon}\right)\left(\frac{1 - \varepsilon + \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{6} + \ \ldots}{\left(1 - \frac{\varepsilon}{2} + \frac{\varepsilon^2}{6} + \ \ldots\right)}\right) \tag{11.17}$$

$$= \frac{-\partial}{\partial\varepsilon}\left(\frac{1}{\varepsilon}\right)\left(1 - \varepsilon + \frac{\varepsilon^2}{2} - \cdots\right)\left(1 + \frac{\varepsilon}{2} - \frac{\varepsilon^2}{6} + \frac{\varepsilon^2}{4} + \ \ldots\right)$$

We have dropped cubic terms and higher. Because ε is small, we can ignore them.

Almost there. We collect terms in (11.17), multiply through by $\frac{1}{\varepsilon}$, and take the derivative. The final result,

$$E = \frac{1}{\varepsilon^2} - \frac{1}{12} \tag{11.18}$$

The first term

$$\frac{1}{\varepsilon^2}$$

is a constant. Since we are concerned only with energy <u>differences</u>, e.g. the difference between the vacuum (ground state) energy and the first energy level in the string spectrum, we can ignore it.

We have already shown that the first state in the spectrum of an open string has energy

$$E = -1 \tag{11.19}$$

Our calculated

$$E = -\frac{1}{12} = -\frac{2}{24} \tag{11.20}$$

in the boosted light-cone system is divided between two dimensions, x and y.

How can we account for the remainder

$$-\frac{22}{24} \tag{11.21}$$

units of energy? It must be distributed in 22 other dimensions! There are twenty-four spatial dimensions in our boosted system. Add the t and z dimensions, and open strings exist in 26 dimensions total.

Return to Table of Contents