We have derived a scattering amplitude for string interactions, so we can describe what happens when strings bump into each other. Now we enter the realm of topology. The scattering amplitude is of a form that requires the world sheets of strings to live on a particular space, Ricci flat space. That is, the mathematical form of the scattering amplitude determines the manifold on which the strings exist, their background spacetime. And, wonder of wonders, that manifold is the spacetime of General Relativity!

Conformal invariance

We derived the Susskind (Polyakov) scattering amplitude in Chapter 14 from sum-over-paths arguments. Now we will re-interpret that amplitude as topology. We seek functions that reproduce the topology of string interactions in simpler form.

Here's the plan. We will show that the argument in the exponent of the Susskind amplitude is conformally invariant. That is, the amplitude describes string interactions on topologies that can be stretched and distorted, so long as local angles are preserved. That allows us to map the string world sheets onto topologies where the amplitudes are easier to calculate.

It's kind of a winding road. In outline, here's what lies ahead.

1. We'll motivate the argument in pictures. Imagining the string world sheet is a stretchy membrane, we'll tug and pull and mold it into a simpler figure.
2. Then we'll derive the conditions (conformal invariance) by which the relations between local coordinates are preserved when we distort the world sheet.
3. Then we'll demonstrate that those conditions are met in the argument of the Susskind amplitude. So string theory is conformally invariant, and we can study string interactions on simpler topologies.

Just to review our argument from the previous chapters: we started with the Feynman diagrams of the T- and S-channels.

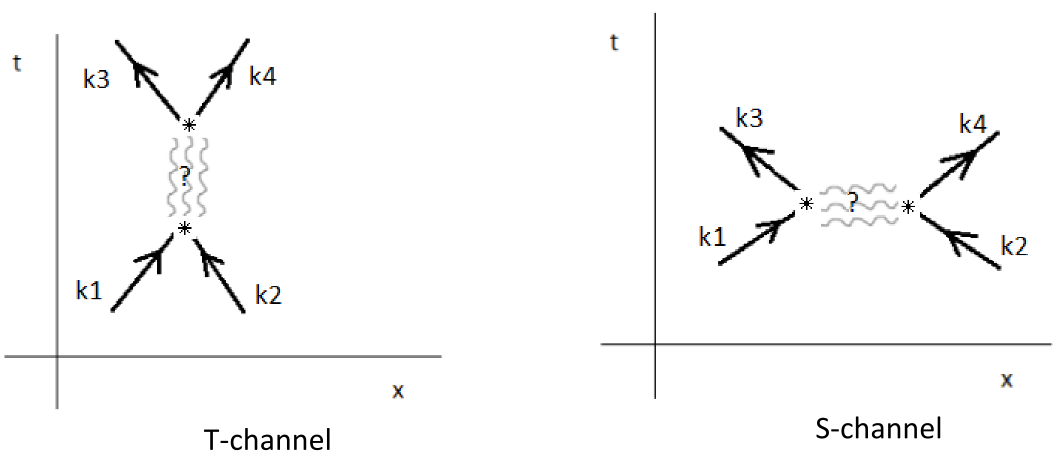

Figure 16.1.

We argued that T- and S- channels can be represented as the world sheet of interacting strings.

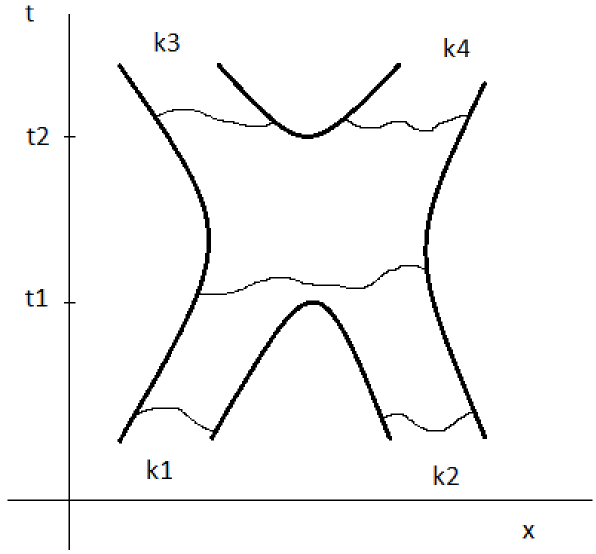

Figure 16.2. Interaction between two open strings. k's represent momenta.

The same interaction in string (τ, σ) coordinates shows the ends of two interacting strings connecting (new "spring" between them) then separating.

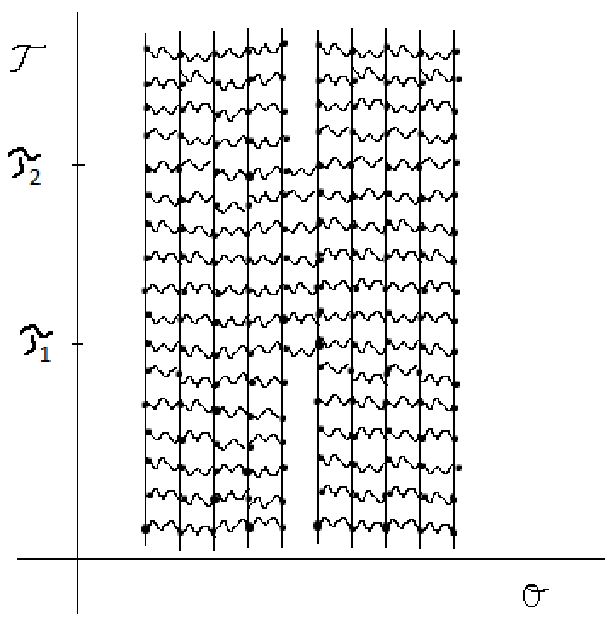

Figure 16.3. Same interaction between two open strings as in Figure 16.2 but drawn in string parameters, σ and τ .

This representation of interacting strings is equivalent in topology to a distorted "bull's eye" (figure below left). We show that we can map this bull's eye world sheet to a disk. The function represented by the disk will provide a straightforward integral to calculate the scattering amplitude. (That's next chapter.)

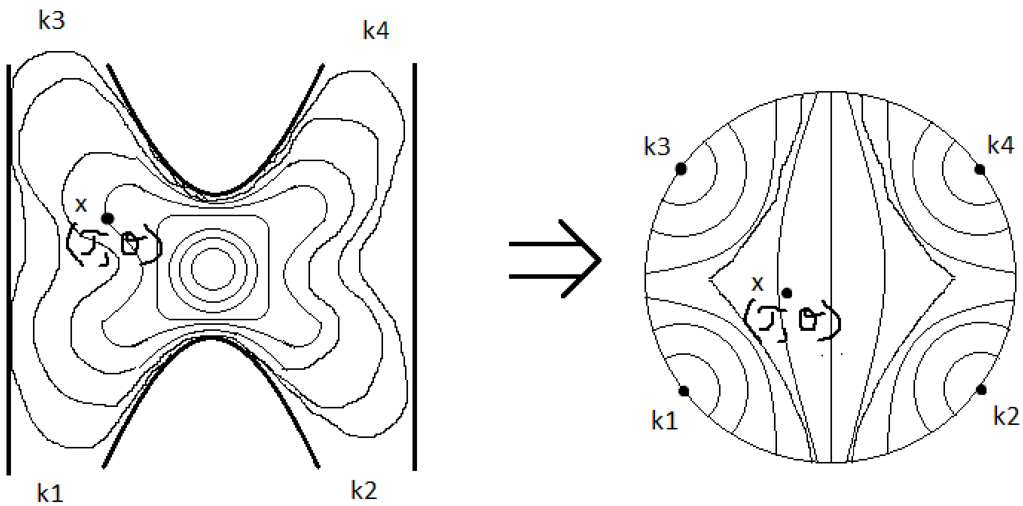

Figure 16.4. Equivalent representations of the topology of the world sheet showing the interaction of two open strings. World sheet representation on the left can be distorted smoothly to produce the disk on the right.

Next we determine the conditions which preserve relations between coordinates while mapping the world sheet from one topology to another, as in Figure 16.4. Let w be a complex function of z. The figure below maps one point in z to the corresponding point in w. Also shown are the corresponding differentials.

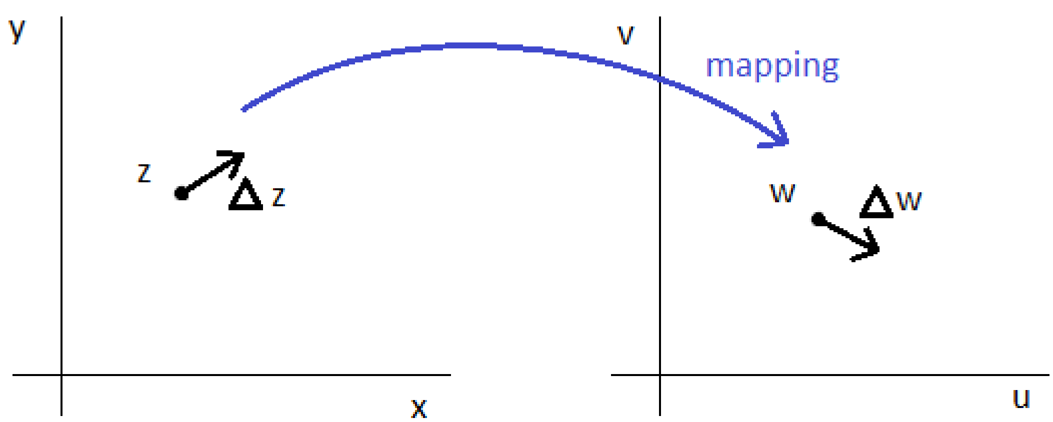

Figure 16.5.

We work in the complex plane, where $z = x + iy$ and $w = u + iv$. To preserve the topology, we require that

$$\frac{\Delta w}{\Delta z} = \frac{du + idv}{dx + idy} \tag{16.1}$$

and this relation does not depend on direction. The equation holds true, for example, whether we take a derivative of z along x, holding y constant or vice versa along y holding x constant.

Along x

$$\frac{\Delta w}{\Delta z} = \frac{du + idv}{dx} = \frac{du}{dx} + \frac{idv}{dx} \tag{16.2}$$

and along y

$$\frac{\Delta w}{\Delta z} = \frac{du + idv}{idy} = \frac{du}{idy} + \frac{idv}{idy} \tag{16.3}$$

Applying the condition of directional independence, we separate the real and imaginary components of these two equations.

$$\frac{du}{dx} = \frac{dv}{dy} \tag{16.4}$$

and

$$\frac{idv}{dx} = \frac{du}{idy} \implies \frac{du}{dy} = -\frac{dv}{dx} \tag{16.5}$$

These are the Cauchy-Riemann equations, named after the mathematicians who found that the equations hold for all analytic functions. If we take the second derivatives of these equations with respect to x and then with respect to y, after a bit of algebra we recover the Laplace equation.

$$\frac{d^2x}{d\tau^2} + \frac{d^2x}{d\sigma^2} = 0 \tag{16.6}$$

Following the logic in reverse, we have shown that functions that satisfy the Laplace equation are conformally invariant. That is, their derivatives – and therefore topological relations – are unchanged by the mapping.

There's more. We show next that angles, also, are preserved by the mapping.

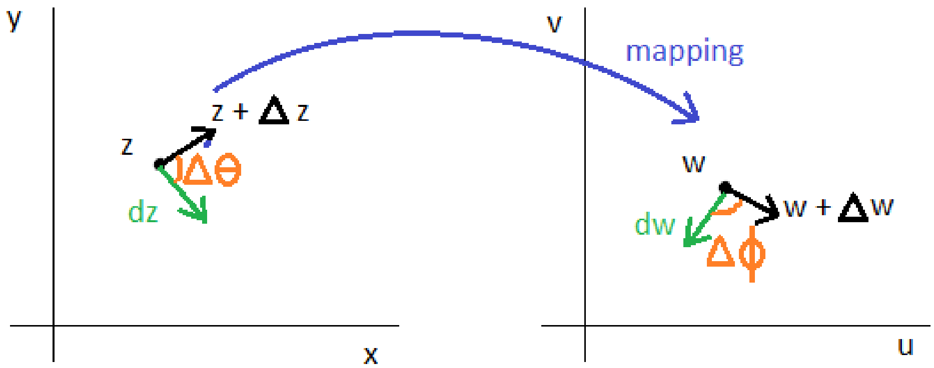

Figure 16.6. Mapping from z to w showing displacement vectors Δz and dz (Δw and dw) separated by angles $\Delta\theta$ ($\Delta\phi$) .

We can re-write Δz and Δw in polar form.

$$\Delta z = \rho e^{i\theta_1} \tag{16.7}$$

and

$$dz = \rho e^{i\theta_2} \tag{16.8}$$

so that

$$\frac{\Delta z}{dz} = \frac{\rho e^{i\theta_1}}{\rho e^{i\theta_2}} = e^{i(\theta_1 - \theta_2)} \tag{16.9}$$

Similarly

$$\frac{\Delta w}{dw} = e^{i(\phi_1 - \phi_2)} \tag{16.10}$$

We have already shown that

$$\Delta z = \Delta w \tag{16.11}$$

by conformal invariance. By similar argument

$$dz = dw \tag{16.12}$$

So

$$\frac{\Delta z}{dz} = \frac{\Delta w}{dw} \tag{16.13}$$

Therefore,

$$e^{i(\theta_1-\theta_2)} = e^{i(\phi_1-\phi_2)} \tag{16.14}$$

The transformation preserves angles.

We have learned that these (conformally invariant) functions obey the Laplace equation and that they preserve angles. Turned around, functions built on the Laplace equations must be conformally invariant.

Now recollect the form of the scattering amplitude. The string scattering amplitude is built on the Laplace equation. It's in the exponential of (14.2), rewritten here.

$$A = \int_{all\ surfaces} e^{-\int_{all\ modes} d\tau d\sigma\left(\left(\frac{\partial x}{\partial \tau}\right)^2+\left(\frac{\partial x}{\partial \sigma}\right)^2\right)} \tag{16.15}$$

That's Laplace, in green.

The string amplitude includes Laplace's equation in its argument. Laplace's equation is conformally invariant, so we have established that the string amplitude is conformally invariant. Hence, we can map the interaction world sheet to a topology that will make it easier to calculate interaction amplitudes. That's what we'll do next.

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