Conformal invariance redux

Here's another way to think about conformal invariance. Understanding conformal invariance helps us figure out the scattering amplitude and also helps us determine what kinds of topologies are allowed in string theory. So it's worth another look.

The Laplace equation appears in the string scattering amplitude (the whole reason why we're interested in it here), and its presence requires that the string world sheet is conformally invariant. Let's dissect Laplace from another angle.

If spacing between points on a string is sufficiently small,

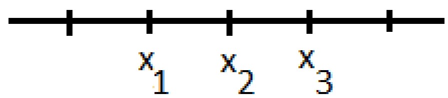

Figure 17.1. Points on a string.

then the first derivative is given by

$$\frac{dx}{d\sigma} = (x_3 - x_2) \tag{17.1}$$

and the second derivative is

$$\frac{d^2x}{d\sigma^2} = (x_3 - x_2) - (x_2 - x_1) = x_3 + x_1 - 2x_2 \tag{17.2}$$

In two dimensions, as on the world sheet,

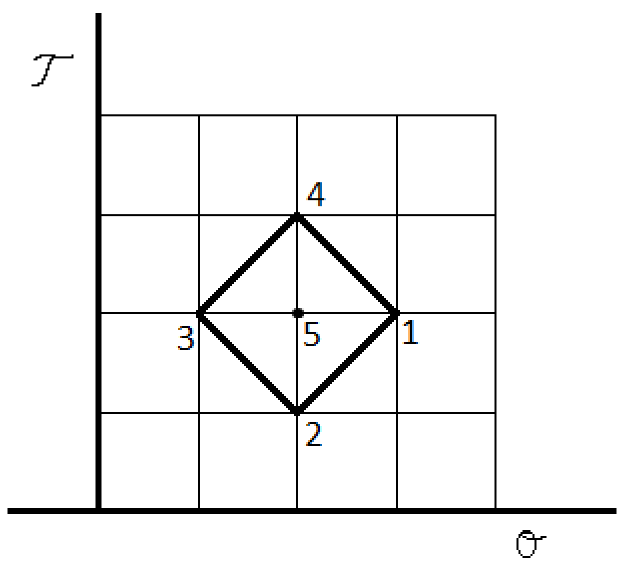

Figure 17.2. World sheet, labeled with string parameters.

$$\frac{d^2x}{d\tau^2} = x_4 + x_2 - 2x_5 \tag{17.3}$$

and

$$\frac{d^2x}{d\sigma^2} = x_1 + x_3 - 2x_5 \tag{17.4}$$

By the Laplace equation, the sum of these derivatives must equal zero, so

$$\frac{d^2x}{d\tau^2} + \frac{d^2x}{d\sigma^2} = x_4 + x_2 - 2x_5 + x_1 + x_3 - 2x_5 = 0 \tag{17.5}$$

Therefore,

$$4x_5 = x_1 + x_2 + x_3 + x_4 \tag{17.6}$$

Put another way, the value at the central point is the average of the corners of the surrounding square. This relation holds independently of the orientation of the little square. And, if the square is small enough, the relation holds even if the larger surface is stretched and distorted.

Equivalently, any mapping that preserves angles preserves this relation between the center and its neighbors. So any mapping that preserves angles preserves the Laplace equation. And any mapping that preserves the Laplace equation guarantees that the string action is minimized.

We can re-map a complicated surface, like the world sheet representing string interactions, to simplify our analysis and allow us to calculate the amplitude for an interaction. As long as the tiny squares are preserved – or, equivalently, as long as angles are preserved – the amplitude stays the same when we stretch or relax the world sheet.

Return to Table of Contents