Closed strings live on Ricci-flat surfaces

We showed that the Laplace equation in the argument of the string amplitude requires that the topology of strings obeys conformal invariance. We extend that argument now to show that the surfaces on which closed strings live must be Ricci-flat. (What's that?) And if Ricci-flat, then string theory requires General Relativity, and vice versa. Quite a payoff!

Here's the argument. We use closed strings, this time.

Closed strings occupy a finite area. We can think of them as a tangle stretched out on the surface. Even in its ground state the string must occupy an extended area because of zero-point oscillations.

Imagine that the string sits on a highly curved surface such as a sphere. (The same argument would hold if the string sat on a roiling surface with hills and valleys at the scale of the string.)

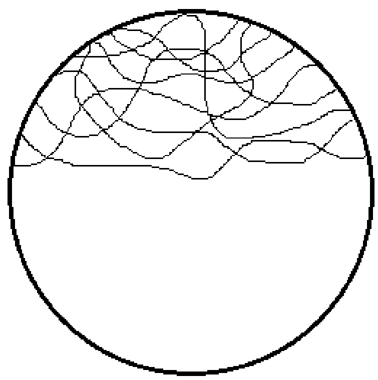

Figure 19.1. Tangle of string on a sphere.

We want to calculate the area of the tangle as a function of mode numbers on the oscillating string. Some necessary background:

1. On a sphere, the string has quantized angular momentum.

$$L = n\hbar \tag{19.1}$$

where n is an integer. (We're counting modes, labeled by n.) As usual we'll set $\hbar = 1$.

2. The kinetic energy of the string is

$$KE = \frac{L^2}{2I} \tag{19.2}$$

where I is the moment of inertia of the string moving around the sphere.

The area of the closed string is calculated from its expectation value.

$$A = \langle 0|x^2|0\rangle = \left\langle 0 \left| \left(\frac{a_n{}^- + a_n{}^+}{\sqrt{n}}\right)\left(\frac{a_m{}^- + a_m{}^+}{\sqrt{m}}\right) \right| 0 \right\rangle \tag{19.3}$$

$$= \sum_{n,m} \left(\frac{a_n{}^- + a_n{}^+}{\sqrt{n}}\right)\left(\frac{a_m{}^- + a_m{}^+}{\sqrt{m}}\right) \cos(n\theta)\cos(m\theta)$$

We've seen these sums before. The product of cosines disappears unless $n = m$, in which instance $cos^2(n\theta) = \frac{1}{2}$. So

$$A = \frac{1}{2}\sum_{n}^{n_{max}} \frac{1}{n} = \frac{1}{2}\ln(n) \tag{19.4}$$

String area increases as the number of modes increases. Draped on a sphere, the string tangle creeps over the surface, pole to pole. As it does, the moment of inertia around the center of the sphere decreases. (The lever arm of the moment becomes symmetric around the sphere.)

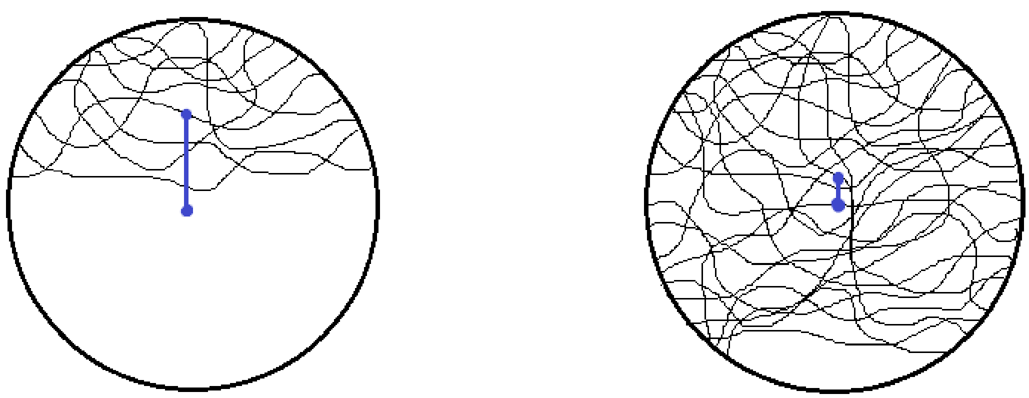

Figure 19.2. Moments of inertia of tangles on spheres. On left, relatively large moment of inertia for tangle that covers large portion of upper hemisphere. On right, small moment for tangle covering most of the sphere. Dots show center of mass of the sphere and c.m. of the string.

And as the moment of inertia approaches zero, the energy blows up.

$$\frac{L^2}{2I} \to \infty \;\; as \;\; I \to 0 \tag{19.5}$$

Not allowed. So strings cannot live on spheres (or other highly curved surfaces at the scale of the strings). They live on flat surfaces, specifically Ricci-flat topologies.

What is “Ricci-flat?” Enter general relativity.

Out there in spacetime, we measure gravity as geometry. Ignoring constant coefficients

$$G \approx T \tag{19.6}$$

The geometry (curvature), G, of spacetime depends on the distribution of mass (energy) and momentum, T. These are both tensors, measuring changes for example in the x curvature as we move along $x, y, z,$ and t.

We have discovered that closed strings live in flat space. G itself has two components in that space.

$$G = R_{\mu\upsilon} - \frac{1}{2} g_{\mu\upsilon} R_{\alpha}^{\alpha} = 0 \tag{19.7}$$

Einstein’s field equation for flat space

$R_{\mu\upsilon}$, the Ricci curvature tensor, is an inherent property of the spacetime. It determines, in the absence of energy and momentum, whether the intrinsic spacetime is curved or flat. R_{α}^{α}, the curvature scalar, sets the scale for the coordinates on which the metric, $g_{\mu\upsilon}$, acts. $g_{\mu\upsilon}$ tells spacetime how to curve, i.e. tells meter sticks how to bend and stretch, tells clocks how to speed up or slow down, as you move along the various axes. (Near a black hole, for example, the Schwarszchild metric applies.) In flat space (the realm of the strings) $\partial g_{\mu\upsilon} = 0$, i.e. the metric does not change. That’s not quite enough, though, to satisfy the field equation for flat space. $g_{\mu\upsilon}$ might everywhere have the same value. So both $R_{\mu\upsilon}$ and R_{α}^{α} must equal zero. This is Ricci-flat space.

Turns out there are many topologies that are Ricci-flat – on the order 10^{500} of them. So there are many possible string theories, and lots of room for string theorists to play! Not all of them describe our world. We’re still looking for the ones that do. Some may describe other universes. For purposes of illustration, we will explore strings on the Ricci-flat torus. Most modern theories model strings on Ricci-flat Calabi-Yau spaces.

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