Spring model for the string

Onward. We extend our model for string energy, modeling strings as stretched springs connecting point masses. (Careful here. We're talking about strings with a "t" built from springs with a "p.") For the moment, we consider open strings.

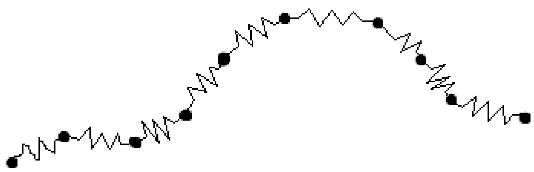

Figure 3.1

Consider the classical spring equation,

$$E = \frac{1}{2}\left(m\left(\frac{dx}{dt}\right)^2 + k(\Delta x)^2\right) \tag{3.1}$$

where m is mass of one of the point masses along the string, k is the spring constant, and Δx measures how much a spring has been stretched between successive masses.

Since we are outside observers embedded in familiar spacetime and measuring strings projected onto that coordinate system, we use good ol' Cartesian variables, x and t. But we are, after all, measuring strings, so the position of a mass on the string at position x on the Cartesian grid is a function of its position, σ, on the string. And the Cartesian measure of how much a spring is stretched is a function of how much the internal coordinates are stretched: $\Delta x = \left(\frac{dx}{d\sigma}\right)\Delta\sigma$.

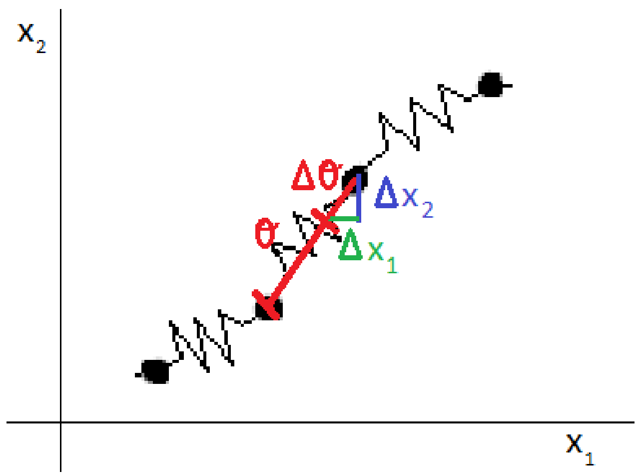

Figure 3.2. Relation between differential string parameter $\Delta\sigma$ and target space coordinates. Dimensions higher than 2 are suppressed.

Some further housekeeping:

- If we divide the string into n segments each with length $\Delta\sigma$, then $n = \frac{\pi}{\Delta\sigma}$, where π is the length of the string.
- By convention we choose $k = \frac{n}{\pi^2} = \frac{1}{\pi\Delta\sigma}$. The logic of this assignment takes some thought. The idea is that spring tension, measured by k, is greater for short springs than long springs. Think of a slinky. It's real easy to grab two ends of a long slinky and stretch it; it's harder to stretch a short segment.

Plugging these relations into (3.1),

$$E = \frac{1}{2}\left(m\left(\frac{dx}{dt}\right)^2 + \frac{1}{\pi\Delta\sigma}\left(\frac{dx}{d\sigma}\right)^2 \Delta\sigma^2\right) \tag{3.2}$$

Simplifying, we get the classical equation for energy of a point mass oscillator in the string.

$$E = \frac{1}{2}\left(m\left(\frac{dx}{dt}\right)^2 + \frac{1}{\pi}\left(\frac{dx}{d\sigma}\right)^2 \Delta\sigma\right) \tag{3.3}$$

One last bit of finagling. We go to the continuous limit, where the number of masses and segments increases indefinitely, so n goes to infinity, $\Delta\sigma$ shrinks toward zero, and we must measure mass in terms of mass density. For convenience, let mass along the string be of unit density, i.e. $\mu = \frac{1}{\Delta\sigma}$, and let each component point mass equal one unit . Then

$$m = \mu\Delta\sigma \tag{3.4}$$

and the energy associated with a point mass

$$E_m = \frac{1}{2}\left(\mu\Delta\sigma\left(\frac{dx}{dt}\right)^2 + \frac{1}{\pi}\left(\frac{dx}{d\sigma}\right)^2 \Delta\sigma\right) \tag{3.5}$$

To find the total internal energy of the string, we sum these terms, including the kinetic and potential components of all the point masses from one end of the string to the other. Total internal energy of the open string is

$$E_T = \frac{1}{2}\int_0^{\pi} d\sigma\left(\mu\left(\frac{dx}{dt}\right)^2 + \frac{1}{\pi}\left(\frac{dx}{d\sigma}\right)^2\right) \tag{3.6}$$

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