Open string modes

We have built a model of strings as spring-and-mass systems. We will use the model to find the string spectrum, first for open strings then closed.

Spectrum? What spectrum?

What distinguishes one string from another is the wave pattern on the string. Strings can oscillate in any of a number of discrete modes. We are interested whether the different modes reproduce the characteristics of the familiar particles of the standard model.

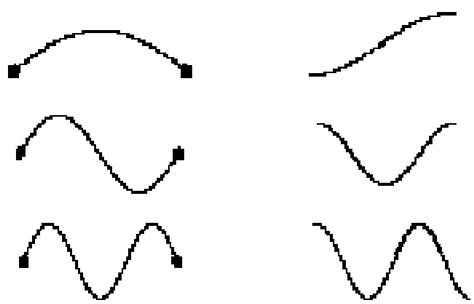

Figure 5.1.   First three modes of an open string, Dirichlet strings on the left, fixed to D0 branes, and Neumann strings on the right. Credits Penn State University Physics.

We study the x and y motion of strings boosted along the z-axis. We shall assume the rules of field theory apply: we can build strings by Fourier composition of different modes.

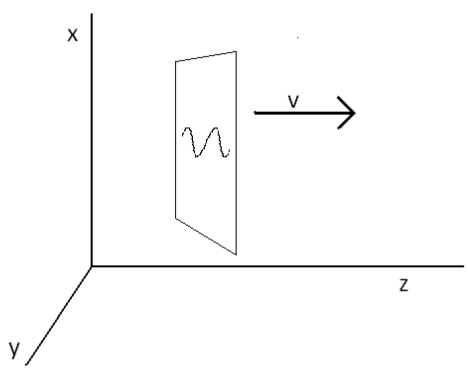

Figure 5.2. Open string boosted along the z-axis.

For the moment, we consider Neumann open strings. We'll need a separate analysis for Dirichlet strings, when we study branes, and another model for closed strings. For Neumann strings, we write the equation for string amplitude (displacement from rest position of each point along the string) as a function of the string parameters, σ and τ .

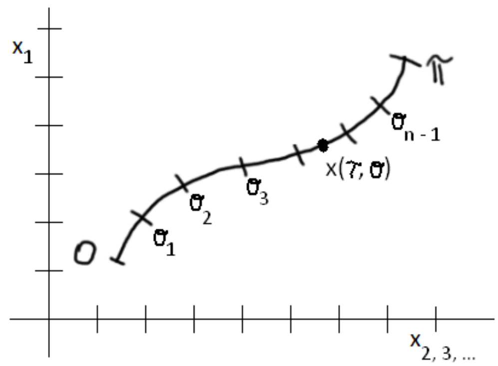

Figure 5.3. Cartesian coordinate x as a function of string parameters (σ, τ). The tau parameter is not shown on the graph.

The x position of a point (τ, σ) on the string is the Fourier sum of the contributions of excited modes.

$$x(\tau, \sigma) = \sum_{n=0}^{\infty} x_n(\tau) \, cos(n\sigma) \tag{5.1}$$

x_n is the amplitude associated with the n^{th} mode. In the cos term, n plays the role of the wavenumber, $\frac{2\pi}{\lambda}$, of that mode. For full description of the wave pattern, we would include a similar equation for the y coordinate of each point on the string. In our discussion, we'll concentrate on x. We would repeat all the calculations for y.

Start with the classical string energy

$$E = \frac{1}{2\pi} \int_0^{\pi} d\sigma \left(\mu \left(\frac{dx}{d\tau} \right)^2 + k \left(\frac{dx}{d\sigma} \right)^2 \right) \tag{5.2}$$

We shall find the energy as a function of the modes of oscillation in (5.1). Substitute first for the kinetic energy term in (5.2), $\mu \left(\frac{dx}{d\tau} \right)^2$. We assign $\mu = 1$ for convenience.

The only time-dependent term on the right of (5.1) is $x(\tau)$. So

$$\frac{dx}{d\tau} = \dot{x} = \sum_{n=0}^{\infty} \dot{x} \; cos(n\sigma) \tag{5.3}$$

and

$$\dot{x}^2 = \sum_{n=0,\; m=0}^{\infty} \dot{x}_n \, \dot{x}_m \int_0^{\pi} d\sigma \left(cos(n\sigma) \, cos(m\sigma) \right) \tag{5.4}$$

Some explanation. We need the double sum because each term in $\dot{x}^2$ has to be multiplied by every other term in the series. Looks like an awful mess, but we're in luck. It turns out that the integral evaluates to zero for all $n \neq m$. When $n = m = 0$, the integral evaluates to π, and for all other $n = m$ the integral evaluates to $\frac{\pi}{2}$. We find that the kinetic energy of the string is

$$\dot{x}^2 = \sum_{n=0,\ m=0}^{\infty} \dot{x}_n \dot{x}_m \int_0^{\pi} d\sigma\big(cos(n\sigma)cos(m\sigma)\big) = \frac{\dot{x}_0{}^2}{2} + \frac{1}{4}\sum_{n=1}^{\infty} \dot{x}_n{}^2 \tag{5.5}$$

Note the subscripts. $\dot{x}_0$is the center-of-mass motion of the string (a string with no excited modes). $\dot{x}_n$contribute the internal kinetic terms, the motions of all the points on the vibrating string.

Next we substitute for the potential term in (5.2), starting with

$$\frac{\partial x}{\partial \sigma} = x' = \frac{\partial}{\partial \sigma}\sum_{n=0}^{\infty} x(\tau)\, cos(n\sigma) = -\sum_{n=0}^{\infty} nx(\tau)\, sin(n\sigma) \tag{5.6}$$

Squaring

$$x'^2 = \sum_{n=0}^{\infty} nx_n m x_m \int_0^{\pi} d\sigma\ \big(sin(n\sigma)\ sin(m\sigma)\big) \tag{5.7}$$

Same rules apply for the sine integral as for the cosine integral, except when $n = m = 0$ the integral evaluates to zero. We find, then

$$x'^2 = \frac{1}{4}\sum_{n=1}^{\infty} n^2 x_n{}^2 \tag{5.8}$$

The full expression of string energy, including kinetic and potential terms, is

$$E = \frac{\dot{x}_0{}^2}{2} + \frac{1}{4}\sum_{n=1}^{\infty} \dot{x}_n{}^2 + \frac{1}{4}\sum_{n=1}^{\infty} n^2 x_n{}^2 \tag{5.9}$$

We have accounted for center of mass kinetic energy, kinetic energy of the points oscillating transverse to the string, and the potential energy due to tension between points. The internal kinetic and potential terms are expressed as functions of the modes of oscillation. In this form, we're ready to quantize the string.

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